

SINGLE RECURRENCE IN ABELIAN GROUPS

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ABSTRACT. We collect problems on recurrence for measure preserving and topological actions of a countable abelian group, considering combinatorial versions of these problems as well. We solve one of these problems by constructing, in $G_2 := \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, a set S such that every translate of S is a set of topological recurrence, while S is not a set of measurable recurrence. This construction answers negatively a variant of the following question asked by several authors: if $A \subset \mathbb{Z}$ has positive upper Banach density, must $A - A$ contain a Bohr neighborhood of some $n \in \mathbb{Z}$?

We also solve a variant of a problem posed by the author by constructing, for all $\varepsilon > 0$, sets $S, A \subseteq G_2$ such that every translate of S is a set of topological recurrence, $d^*(A) > 1 - \varepsilon$, and the sumset $S + A$ is not piecewise syndetic. Here d^* denotes upper Banach density.

1. INTRODUCTION

1.1. Recurrence in dynamics. A set $S \subseteq \mathbb{Z}$ is a *set of measurable recurrence* if for every measure preserving transformation $T : X \rightarrow X$ of a probability measure space (X, μ) and every $D \subseteq X$ having $\mu(D) > 0$, there exists $n \in S$ such that $\mu(D \cap T^{-n}D) > 0$. We say that S is a *set of topological recurrence* if for every minimal topological system (X, T) , where X is a compact metric space, and $U \subseteq X$ is a nonempty open set, there exists $n \in S$ such that $U \cap T^{-n}U \neq \emptyset$. Every set of measurable recurrence is also a set of topological recurrence, since every minimal topological system admits an invariant probability measure of full support. The concepts of measurable and topological recurrence generalize to actions of abelian groups; see Section 2.1 for definitions.

Vitaly Bergelson asked if there is a set $S \subseteq \mathbb{Z}$ which is a set of topological recurrence but not a set of measurable recurrence, and in [30] Igor Kříž constructed such a set. Alan Forrest, in his doctoral thesis [14], produced an analogous example for actions of $G_2 := \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.

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Our main result, Theorem 1.1, is a more robust example: we find a set $S \subseteq G_2$ such that every translate of S is a set of topological recurrence while S is not a set of measurable recurrence.

If G is an abelian group and $A, B \subseteq G$, $g \in G$, we write $A + B$ for the *sumset* $\{a + b : a \in A, b \in B\}$, $A - B$ for the *difference set* $\{a - b : a \in A, b \in B\}$, and $A + g$ for the *translate* $\{a + g : a \in A\}$.

Theorem 1.1. *There is a set $S \subseteq G_2$ such that for all $g \in G_2$, $S + g$ is a set of topological recurrence, while S is not a set of measurable recurrence.*

If (X, μ, T) is a measure preserving G -system and $D \subseteq X$ has $\mu(D) > 0$, the set $\text{Ret}_T(D) := \{g \in G : \mu(D \cap T^g D) > 0\}$ is of interest. One way to study such sets is to identify sets of measurable recurrence. Part (iii) of Lemma 2.12 and Lemma 5.8 show that Theorem 1.1 is equivalent to the following statement: there is an ergodic measure preserving G_2 -system (X, μ, T) and $D \subseteq X$ with $\mu(D) > 0$ such that $\text{Ret}_T(D)$ does not contain a set of the form $g + (B - B)$ where $B \subseteq G_2$ is piecewise syndetic and $g \in G_2$. See Section 2.3 for the definition of “piecewise syndetic.”

1.2. Difference sets. If G is a countable abelian group and $A \subseteq G$, let $d^*(A)$ denote the upper Banach density of A ; see Sections 2.4 and 2.7 for definitions of Bohr neighborhoods and upper Banach density. A theorem of Følner [12] states that when $d^*(A) > 0$, $A - A$ contains a set of the form $U \setminus Z$, where U is a Bohr neighborhood of $0 \in G$ and $d^*(Z) = 0$. For $G = \mathbb{Z}$, Kríž’s construction [30] exhibits a set A having $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of 0, and in fact $A - A$ does not contain a set of the form $S - S$, where $S \subseteq \mathbb{Z}$ is piecewise syndetic. Several authors have asked ([5, 16, 20, 24]) whether $A - A$ must contain a Bohr neighborhood of *some* $n \in \mathbb{Z}$ whenever $d^*(A) > 0$, and this question remains open. For $G_p := \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$, where p is prime, the author constructed in [23] sets $A \subseteq G_p$ having $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of *any* $g \in G_p$.

From a combinatorial perspective, the sets $\text{Ret}_T(D)$ are essentially difference sets $A - A$, where $d^*(A) > 0$ (as shown by Lemma 5.8). We prove Theorem 1.1 by improving the result of [23] in the special case $p = 2$, with a simpler proof than in [23]. The following theorem summarizes the result of our construction.

Theorem 1.2. *For all $\varepsilon > 0$, there is a set $A \subseteq G_2$ such that $d^*(A) > \frac{1}{2} - \varepsilon$, and $A - A$ does not contain a set of the form $B - B + g$, where $B \subseteq G_2$ is piecewise syndetic and $g \in G_2$.*

Theorems 1.1 and 1.2 are proved in Sections 3 and 4, which are mostly self-contained. The proof of Theorem 1.2 is a combinatorial construction requiring no background regarding dynamical systems.

Remark 1.3. When C is a Bohr neighborhood of 0 in a countable abelian group G , it contains a difference set $B - B$ where B is syndetic (see Remark 2.6), so the present Theorem 1.2 implies the special case of [23, Theorem 1.2] where $p = 2$. However, we do not know whether piecewise syndeticity of $B \subseteq G_2$ implies $B - B$ contains a Bohr neighborhood – this is equivalent to Part (i) of Question 2.2 in the case $G = G_2$. So we cannot be certain that Theorem 1.2 is a strict improvement over [23, Theorem 1.2].

In light of a correspondence principle (see Lemma 5.8), we may also state Theorem 1.2 as a more quantitative form of Theorem 1.1.

Theorem 1.4. *For all $\varepsilon > 0$, there is*

- *a measure preserving action T of G_2 on a probability measure space (X, μ) ,*
- *a set $D \subseteq X$ such that $\mu(D) > \frac{1}{2} - \varepsilon$, and*
- *a set $S \subseteq G_2$ such that every translate of S is a set of topological recurrence,*

such that $D \cap T^g D = \emptyset$ for all $g \in S$.

1.3. Sumsets. If G is a countable abelian group and $A, B \subseteq G$ have positive upper Banach density, then $A + B$ is piecewise syndetic. For $G = \mathbb{Z}$, this result is due to Renling Jin [26]. Bergelson, Furstenberg, and Weiss [4] strengthened the conclusion from “piecewise syndetic” to “piecewise Bohr” (see Section 2.5). In [20] and [19], the author generalized these results to cases where $d^*(A) = 0$ and $d^*(B) > 0$, and these results have been generalized to other settings – see [1, 2, 6, 10, 11]. The proofs and examples in [20] raised the following question, stated for $G = \mathbb{Z}$ as [20, Question 5.1].

Question 1.5. Let G be a countable abelian group and $S \subseteq G$. Let \tilde{S} be the closure of S in bG , the Bohr compactification¹ of G . Let m_{bG} denote Haar measure in bG . Which, if any, of the following implications are true?

¹We will not use the Bohr compactification in this article except in reference to Question 1.5 – see Section 2.6 for a brief discussion.

- (1) If $m_{bG}(\tilde{S}) > 0$ and $d^*(A) > 0$, then $S + A$ is piecewise syndetic.
- (2) If $m_{bG}(\tilde{S}) > 0$ and $d^*(A) > 0$, then $S + A$ is piecewise Bohr.
- (3) If S is dense in the Bohr topology of G and $d^*(A) > 0$, then $d^*(S + A) = 1$.

Theorem 1.4 of [23] provides counterexamples to implications (2) and (3) when $G = G_p := \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ for some prime p . The construction in the proof of the present Theorem 1.2 provides counterexamples to all three implications for $G = G_2$, resulting in the following theorem. All parts of Question 1.5 remain open for all countably infinite abelian groups besides G_p for some prime p .

Theorem 1.6. *For all $\varepsilon > 0$, there are sets $S, A \subseteq G_2 := \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ such that $d^*(A) > 1 - \varepsilon$, every translate of S is a set of topological recurrence, and $S + A$ is not piecewise syndetic.*

We now explain why Theorems 1.1 and 1.4 let us resolve Question 1.5 for $G = G_2$. This explanation uses the notion of “Bohr recurrence” from Definition 2.1.

Lemma 5.3, together with the straightforward implication “(S is a set of topological recurrence) \implies (S is a set of Bohr recurrence)” shows that if every translate of S is a set of topological recurrence, then S is dense in the Bohr topology of G_2 . The condition that S is dense in the Bohr topology of G_2 implies $\tilde{S} = bG_2$ in the notation of Question 1.5. In particular, $m_{bG_2}(\tilde{S}) = 1$, so Theorem 1.6 provides counterexamples to all parts of Question 1.5 for $G = G_2$.

The next section surveys some notions of recurrence for dynamical systems and formulates some questions related to Theorems 1.1 and 1.2. Section 3 introduces the definitions and notation needed for our constructions, and Section 4 contains the proofs of Theorems 1.1, 1.2, 1.4, and 1.6. Section 5 contains some standard lemmas needed to relate statements about difference sets to statements about dynamical systems. Section 5 contains some standard lemmas needed to keep the article self-contained. These are mostly for exposition, with the exception of Lemma 5.8, which is used only to derive Theorems 1.1 and 1.4 from the proof of Theorem 1.2. Sections 3 and 4 can be read essentially independently of the others.

2. TWO HIERARCHIES OF SINGLE RECURRENCE PROPERTIES

2.1. Measurable, topological, and Bohr recurrence. We begin with some standard definitions.

Let G be a countable abelian group.

A *measure preserving G -system* (or briefly, “ G -system”) is a triple (X, μ, T) , where (X, μ) is a probability measure space and T is an action of G on X preserving μ , meaning $\mu(T^g D) = \mu(D)$ for every measurable $D \subseteq X$ and $g \in G$. We say that (X, μ, T) is *ergodic* if for all measurable $D \subseteq X$ satisfying $\mu(D \triangle T^g D) = 0$ for all $g \in G$, $\mu(D) = 0$ or $\mu(D) = 1$.

A *topological G -system* is a pair (X, T) , where X is a compact metric space and T is an action of G on X by homeomorphisms. We say that (X, T) is *minimal* if for all $x \in X$, the orbit $\{T^g x : g \in G\}$ is dense in X .

A *group rotation G -system* is a pair (Z, R_ρ) where Z is a compact abelian group, $\rho : G \rightarrow Z$ is a homomorphism, and $R_\rho^g z = z + \rho(g)$ for $z \in Z$ and $g \in G$. Such a system (Z, R_ρ) may be considered as a topological G -system, or as a measure preserving G -system (Z, m, R_ρ) , where m is normalized Haar measure on Z . The topological G -system (Z, R_ρ) is minimal if and only if the measure preserving G -system (Z, m, R_ρ) is ergodic if and only if $\rho(G)$ is dense in Z .

Definition 2.1. We say that a set $S \subseteq G$ is a

- *set of measurable recurrence* if for all measure preserving G -systems (X, μ, T) and every $D \subseteq X$ having $\mu(D) > 0$, there exists $g \in S$ such that $\mu(D \cap T^g D) > 0$.
- *set of strong recurrence* if for all measure preserving G -systems (X, μ, T) and every $D \subseteq X$ having $\mu(D) > 0$, there exists $c > 0$ such that $\{g \in S : \mu(D \cap T^g D) > c\}$ is infinite.
- *set of optimal recurrence* if for all measure preserving G -systems (X, μ, T) , every measurable $D \subseteq X$ and $c < \mu(D)^2$, there is a $g \in S$ such that $\mu(D \cap T^g D) > c$.
- *set of topological recurrence* if for every minimal topological G -system (X, T) and every open nonempty $U \subseteq X$, there exists $g \in S$ such that $U \cap T^g U \neq \emptyset$.
- *set of Bohr recurrence* if for every minimal group rotation G -system (Z, R_ρ) and every open nonempty $U \subseteq Z$, there exists $g \in S$ such that $U \cap R_\rho^g U \neq \emptyset$.

Let \mathcal{S}^1 be the group $\{z \in \mathbb{C} : |z| = 1\}$, the complex numbers of modulus 1 with the group operation of multiplication. A set $S \subseteq G$ is *equidistributed* if there is a sequence of finite subsets $S_j \subseteq S$ such that for every non-constant homomorphism $\chi : G \rightarrow \mathcal{S}^1$, $\lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \chi(g) = 0$.

We abbreviate the above definitions in the following conditions.

(R_1) S is equidistributed.

- (R_2) S is a set of optimal recurrence.
- (R_3) S is a set of strong recurrence.
- (R_4) S is a set of measurable recurrence.
- (R_5) S is a set of topological recurrence.
- (R_6) S is a set of Bohr recurrence.

We have $(R_i) \implies (R_{i+1})$ for $i = 1, \dots, 5$. These implications are well known – see Section 5.5 for a proof of $(R_1) \implies (R_2)$ and further discussion. We briefly summarize what is known regarding the reverse implications $(R_j) \implies (R_i)$ for $j > i$. Here “group” means “countably infinite abelian group” and G_p denotes $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.

The question of whether $(R_6) \implies (R_5)$ is well known and remains open for every group G . The question was first explicitly asked in this form by Katznelson [28] for $G = \mathbb{Z}$, but the question is older – see [39], as well as [17] and [8] for exposition and a related problem.

For $G = \mathbb{Z}$, Kříž [30] proved that $(R_5) \not\Rightarrow (R_4)$, and Forrest [14] adapted Kříž’s example to $G = G_2$. Randall McCutcheon [32, 33] presented a simplification of Kříž’s example due to Imre Ruzsa. Our proof of Theorem 1.1 provides another proof that $(R_5) \not\Rightarrow (R_4)$ for $G = G_2$. Whether $(R_5) \implies (R_4)$ is open for all groups G besides G_2 and \mathbb{Z} .

For $G = \mathbb{Z}$ and $G = G_2$, Forrest [13] proved that $(R_4) \not\Rightarrow (R_3)$ and $(R_3) \not\Rightarrow (R_2)$. For $G = \mathbb{Z}$, McCutcheon [32] provides a simplification of Forrest’s construction and [21] provides another proof of $(R_4) \not\Rightarrow (R_3)$. The status of the implications $(R_4) \implies (R_3)$ and $(R_3) \implies (R_2)$ is unknown for all other groups G .

For $G = \mathbb{Z}$, the classical example $S = \{2n : n \in \mathbb{Z}\}$ shows that $(R_2) \not\Rightarrow (R_1)$. Constructing examples showing that $(R_2) \not\Rightarrow (R_1)$ for an arbitrary countably infinite abelian group is not difficult, but it makes an interesting exercise.

2.2. Translations. If G is a countable abelian group and $S \subseteq G$, we say that S satisfies property (R_i^\bullet) if every translate of S satisfies property (R_i) , meaning $S + g$ satisfies (R_i) for all $g \in G$. It is easy to verify that $(R_1^\bullet) \iff (R_1)$, while $(R_i) \not\Rightarrow (R_i^\bullet)$ for each $i > 1$ and every nontrivial group G . Observe that $(R_i^\bullet) \implies (R_j)$ if and only if $(R_i^\bullet) \implies (R_j^\bullet)$.

We say that S satisfies property (PR_i) if $S \setminus \{0\}$ satisfies property (R_i) .

We summarize what is currently known regarding the implications $(R_i^\bullet) \implies (R_j)$.

For every countable abelian group G , $(R_6^\bullet) \not\Rightarrow (R_1)$. For $G = \mathbb{Z}$ this is due to Katznelson [27, Theorem 2.2], and for general G to Saeki [37], by way of constructing sets $S \subseteq G$ dense in the Bohr topology that do not satisfy (R_1) . See Lemma 5.3 for a proof that such constructions prove $(R_6^\bullet) \not\Rightarrow (R_1)$.

For $G = \mathbb{Z}$, whether the implication $(R_6^\bullet) \Rightarrow (R_4)$ holds has been asked² in [5], [16], [20], and [24]. The problem remains stubbornly open. For $G_p = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$, where p is prime, the author proved in [23] that $(R_6^\bullet) \not\Rightarrow (R_4)$ by exhibiting a set $A \subseteq G_p$ having $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of any $g \in G_p$. See Lemma 2.12 for an explanation of why said construction implies $(R_6^\bullet) \not\Rightarrow (R_4)$.

For $G_2 = \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, Theorem 1.1 of the present article says that $(R_5^\bullet) \not\Rightarrow (R_4)$.

For $G = \mathbb{Z}$, the main result of [21] shows that $(R_4^\bullet) \not\Rightarrow (R_3)$, and in fact there is a set $S \subseteq \mathbb{Z}$ satisfying (R_4^\bullet) such that no translate of S satisfies (R_3) .

The implications listed above are all that are currently known, so the following questions remain. A priori, the answer to each part could be different for different groups G .

- Question 2.2.** (i) Does $(R_6^\bullet) \Rightarrow (R_5)$? (Open for every G .)
(ii) Does $(R_6^\bullet) \Rightarrow (R_4)$? (Open for every G except G_p where p is prime.)
(iii) Does $(R_5^\bullet) \Rightarrow (R_4)$? (Open for every G except G_2 .)
(iv) Does $(R_4^\bullet) \Rightarrow (R_3)$? (Open for every G except \mathbb{Z} .)
(v) Does $(R_3^\bullet) \Rightarrow (R_2)$? (Open for every G .)
(vi) Does $(R_2^\bullet) \Rightarrow (R_1)$? (Open for every G .)

Furthermore, in case $(R_i^\bullet) \not\Rightarrow (R_j)$, is there a set $S \subseteq G$ satisfying (R_i^\bullet) while no translate of S satisfies (PR_j) ?

We expect that for every G , the answers to parts (ii) through (vi) are all “no.” We reserve speculation on part (i), as a negative answer would resolve the difficult question of whether $(R_6) \Rightarrow (R_5)$, while a positive answer would be surprising.

If the answer to a given part does not depend on the group G , it would be interesting to identify a general principle which implies that the answer must be the same for every G .

²In [5], [16], and [24], the question is phrased as “If $A \subseteq \mathbb{Z}$ has positive upper Banach density, must $A - A$ contain a Bohr neighborhood of some $n \in \mathbb{Z}$?” We discuss this form of the question in Section 2.3.

Finally, while our Theorem 1.1 shows that $(R_5^\bullet) \not\Rightarrow (R_4)$ for $G = G_2$, our proof does not provide an example of a set S satisfying (R_5^\bullet) such that no translate of S satisfies (PR_4) . In fact, for the set S constructed in the proof, Lemma 4.3 implies $S + \mathbf{1}$ satisfies (PR_4) – see Section 3 for notation.

As mentioned at the beginning of this subsection, $(R_6^\bullet) \not\Rightarrow (R_1)$ is established for all countable abelian groups G in [37], so Theorem 1.1 is a refinement of that result for the group G_2 .

2.3. Syndeticity and piecewise syndeticity. In this and the following subsections we formulate some definitions needed to interpret Question 2.2 in terms of difference sets. We fix a countable abelian group G for the remainder of this section.

Definition 2.3. A set $A \subseteq G$ is

- *thick* if for every finite $F \subseteq G$, there exists $g \in G$ such that $F + g \subseteq A$,
- *syndetic* if there is a finite set $F \subseteq G$ such that $A + F = G$, meaning G is a union of finitely many translates of A ,
- *piecewise syndetic* if there is a syndetic set $S \subseteq G$ such that for all finite $F \subseteq S$, there exists $g \in G$ such that $F + g \subseteq A$.

Remark 2.4. Our definition of “piecewise syndetic” is not standard, and Lemma 5.6 shows that it is equivalent to the standard one. We use our definition so that it is easy to see that if A is piecewise syndetic, then $A - A$ contains a set of the form $S - S$, where S is syndetic.

Remark 2.5. A set $A \subseteq G$ is thick if and only if $d^*(A) = 1$.

2.4. The Bohr topology. If G is a countable abelian group, the Bohr topology on G is the weakest topology on G such that every homomorphism $\rho : G \rightarrow \mathbb{T}$ is continuous, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the usual topology. A set $S \subseteq G$ satisfies (R_6) if and only if 0 is in the closure of S in the Bohr topology, and S satisfies (R_6^\bullet) if and only if S is dense in the Bohr topology – see Lemmas 5.2 and 5.3 for proofs. The open sets of the Bohr topology are called *Bohr neighborhoods*. The group operation and inversion are both continuous in the Bohr topology, so U is a Bohr neighborhood of $g \in G$ if and only if $U - g$ is a Bohr neighborhood of 0 . See [34] for exposition of the Bohr topology of locally compact abelian groups, including countable discrete groups.

The Bohr topology is the weakest topology making every homomorphism $\rho : G \rightarrow Z$ to a compact abelian group Z continuous.

A neighborhood base for 0 in the Bohr topology on \mathbb{Z} is the collection of sets of the form $\{n : \max_{1 \leq i \leq d} \|s_i n\| < \varepsilon\}$, where $s_i \in \mathbb{R}$, $\varepsilon > 0$, and $\|x\|$ denotes the distance from x to the nearest integer.

For a fixed prime number p , a neighborhood base for 0 in the Bohr topology on $G_p := \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is the collection of finite index subgroups of G_p . A set $U \subseteq G_p$ is open in the Bohr topology if and only if it is a union of cosets of finite index subgroups of G_p .

2.5. Piecewise Bohr sets. A set $S \subseteq G$ is *piecewise Bohr* if there is a nonempty Bohr neighborhood $U \subseteq G$ such that for every finite $F \subseteq U$, there is a $g \in G$ such that $g + F \subseteq S$. Equivalently, a set $S \subseteq G$ is piecewise Bohr if there is a nonempty Bohr neighborhood U of some g and a thick set C such that $U \cap C \subseteq S$ (see Definition 2.3). While we do not need the equivalence of these definitions, a proof can be found in [20] for the case $G = \mathbb{Z}$.

Remark 2.6. Bohr neighborhoods are syndetic, due to the compactness of \mathbb{T}^d for each d , and every Bohr neighborhood of 0 contains a difference set $A - A$ where A is a Bohr neighborhood of 0. It follows that if A is piecewise Bohr, then $A - A$ contains a set of the form $S - S$, where S is syndetic.

2.6. Bohr compactification. The Bohr compactification bG of a locally compact abelian group G is the unique compact abelian group H such that G embeds densely in H (so that G may be identified with a subgroup \tilde{G} of H) and every continuous homomorphism to a compact abelian group $\rho : G \rightarrow Z$ has a unique continuous extension $\tilde{\rho} : H \rightarrow Z$. The Bohr topology on G is the topology G inherits from bG as a subspace. We will not use the Bohr compactification in this article, but Question 1.5 is reproduced from [20], where it is mentioned. See [34] for further exposition of bG .

2.7. Upper Banach density. A *Følner sequence* for an abelian group G is a sequence of finite subsets $\Phi_n \subseteq G$ such that $\lim_{n \rightarrow \infty} \frac{|(\Phi_n + g) \cap \Phi_n|}{|\Phi_n|} = 1$ for every $g \in G$. Every countable abelian group admits a Følner sequence.³

³The standard way to see this is to appeal to some theory of amenable groups: a countable discrete group is amenable if and only if it admits a Følner sequence, and abelian groups are amenable by the Markov-Kakutani fixed point theorem. However, given a specific abelian group, it is usually possible to construct a Følner sequence by hand.

If $\Phi = (\Phi_n)_{n \in \mathbb{N}}$ is a Følner sequence for G and $A \subseteq G$, the *upper density of A with respect to Φ* is $\bar{d}_\Phi(A) := \limsup_{n \rightarrow \infty} \frac{|A \cap \Phi_n|}{|\Phi_n|}$; we write $d_\Phi(A)$ if the limit exists. The *upper Banach density* of A is $d^*(A) := \sup\{\bar{d}_\Phi(A) : \Phi \text{ is a Følner sequence}\}$. Note that for every $A \subseteq G$, there is a Følner sequence Φ such that $d^*(A) = d_\Phi(A)$.

If $(H_n)_{n \in \mathbb{N}}$ is an increasing sequence of finite subgroups of G such that $G = \bigcup_{n=1}^\infty H_n$, then $(H_n)_{n \in \mathbb{N}}$ is a Følner sequence for G .

While upper Banach density is not finitely additive, it enjoys the following weaker property.

Lemma 2.7. *Let G be a countable abelian group, $g \in G$, and $A \subseteq G$. If $A \cap (A + g) = \emptyset$, then $d^*(A \cap (A + g)) = 2d^*(A)$.*

We omit the proof, which is a straightforward application of the relevant definitions.

2.8. Difference sets and recurrence. The study of measurable and topological recurrence is closely tied to the study of combinatorial structure in difference sets $A - A$. After the following definitions, we state Question 2.11 to rephrase Parts (i)-(iii) of Question 2.2 in terms of difference sets.

Fix a countably infinite abelian group G for the remainder of this subsection.

Definition 2.8. We say that $S \subseteq G$ is a

- *set of chromatic recurrence* if for all $k \in \mathbb{N}$ and every partition $G = \bigcup_{j=1}^k A_j$ of G , there is a $j \leq k$ such that $(A_j - A_j) \cap S \neq \emptyset$,
- *set of density recurrence* if for every $A \subseteq G$ having $d^*(A) > 0$, $(A - A) \cap S \neq \emptyset$.

The following proposition is a special case of Theorems 2.2 and 2.6 of [3].

Proposition 2.9. *Let $S \subseteq G$.*

- (i) *S is a set of measurable recurrence if and only if S is a set of density recurrence.*
- (ii) *S is a set of topological recurrence if and only if S is a set of chromatic recurrence.*

We also need the following lemma, which follows immediately from our definition of “piecewise syndetic” and the partition regularity of piecewise syndeticity (Lemma 5.7).

Lemma 2.10. *A set $S \subseteq G$ is a set of chromatic recurrence if and only if $(A - A) \cap S \neq \emptyset$ whenever A is piecewise syndetic.*

The first three parts of Question 2.11 rephrase Parts (i)-(iii) of Question 2.2 in terms of difference sets. Part (iv) rephrases the question of whether $(R_6) \implies (R_5)$ in terms of difference sets.

Question 2.11. Let G be a countable abelian group and $A \subseteq G$.

- (i) Does A being piecewise syndetic imply that $A - A$ contains a Bohr neighborhood of some $g \in G$? (Open for every G .)
- (ii) Does $d^*(A) > 0$ imply that $A - A$ contains a Bohr neighborhood of some $g \in G$? (Open for every G except G_p for prime p , the main result of [23] shows that the answer is “no” for these G_p .)
- (iii) Does $d^*(A) > 0$ imply that $A - A$ contains a set of the form $B - B + g$, where $g \in G$ and $B \subseteq G$ is piecewise syndetic? (Open for every G except G_2 , where Theorem 1.2 provides a negative answer.)
- (iv) Does A being piecewise syndetic imply $A - A$ contains a Bohr neighborhood of $0 \in G$? (Open for every G .)

Interpreting the assertion $(R_4^\bullet) \not\Rightarrow (R_3)$ in terms of difference sets requires more intricate statements than those in Question 2.11, see [21] for details in the case $G = \mathbb{Z}$.

The next lemma proves that Parts (i)-(iii) of Question 2.11 are really equivalent to the corresponding parts of Question 2.2, and that Part (iv) of Question 2.11 is equivalent to the question “does $(R_6) \implies (R_5)$?”. Lemma 2.13 provides a similar reformulation in terms of sets of return times.

Lemma 2.12. *Let G be a countable abelian group and $A \subseteq G$.*

- (i) *(A is piecewise syndetic $\implies A - A$ contains Bohr neighborhood of some $g \in G$) if and only if $(R_6^\bullet) \implies (R_5)$.*
- (ii) *($d^*(A) > 0 \implies A - A$ contains a Bohr neighborhood of some $g \in G$) if and only if $(R_6^\bullet) \implies (R_4)$.*
- (iii) *($d^*(A) > 0 \implies A - A$ contains a set $g + B - B$, where $B \subseteq G$ is piecewise syndetic and $g \in G$) if and only if $(R_5^\bullet) \implies (R_4)$.*
- (iv) *(A is piecewise syndetic $\implies A - A$ contains a Bohr neighborhood of $0 \in G$) if and only if $(R_6) \implies (R_5)$.*

Proof. We prove only (iv). The other equivalences are proved similarly.

First suppose that if $A - A$ contains a Bohr neighborhood of $0 \in G$ whenever A is piecewise syndetic, and that $S \subseteq G$ satisfies (R_6) . Then

$S \cap (A - A) \neq \emptyset$ whenever A is piecewise syndetic, by Lemma 5.2, and S therefore satisfies (R_5) , by Proposition 2.9 and Lemma 2.10.

Now suppose $(R_6) \implies (R_5)$, and that $A \subseteq G$ is piecewise syndetic. Assume, to get a contradiction, that $A - A$ does not contain a Bohr neighborhood of 0. Then $(A - A)^c := G \setminus (A - A)$ has nonempty intersection with every Bohr neighborhood of 0, so that $(A - A)^c$ satisfies (R_6) , by Lemma 5.2. Since we are assuming $(R_6) \implies (R_5)$, we conclude that $(A - A)^c$ satisfies (R_5) , so that Proposition 2.9 and Lemma 2.10 imply $(A - A)^c \cap (A - A) \neq \emptyset$, a contradiction. \square

Let X be a set and T an action of G on X . If $D \subseteq X$ let $\text{Ret}_T(D) := \{g \in G : D \cap T^g D \neq \emptyset\}$. The following lemma provides equivalent formulations of Parts (i)-(iii) of Question 2.2 in terms of the sets $\text{Ret}_T(D)$.

Lemma 2.13. *Let G be a countable abelian group.*

- (i) *(For every minimal G -system (X, T) and every open $U \subseteq X$, $\text{Ret}_T(U)$ contains a Bohr neighborhood of some $g \in G$) if and only if $(R_6^\bullet) \implies (R_5)$*
- (ii) *(For every measure preserving G -system (X, μ, T) and $D \subseteq X$ having $\mu(D) > 0$, the set $\text{Ret}_T(D)$ contains a Bohr neighborhood of some $g \in G$) if and only if $(R_6^\bullet) \implies (R_4)$.*
- (iii) *(For every measure preserving G -system (X, μ, T) and $D \subseteq X$ having $\mu(D) > 0$, the set $\text{Ret}_T(D)$ contains a set of the form $B - B + g$, where B is piecewise syndetic) if and only if $(R_5^\bullet) \implies (R_4)$.*
- (iv) *(For every minimal G -system (X, T) and every open $U \subseteq X$, $\text{Ret}_T(U)$ contains a Bohr neighborhood of 0) if and only if $(R_6) \implies (R_5)$.*

We omit the proof of Lemma 2.13; like Lemma 2.12 it may be proved with straightforward applications of Lemmas 5.2 and 5.3.

2.9. Sumsets and measure expansion.

Definition 2.14. Let G be a countable abelian group and $S \subseteq G$. We say that S is

- *measure expanding*⁴ if for every ergodic measure preserving G -system (X, μ, T) and every $D \subseteq X$ having $\mu(D) > 0$, we have $\mu(\bigcup_{g \in S} T^g D) = 1$.
- *measure transitive* if for every (not necessarily ergodic) measure preserving G -system (X, μ, T) and every pair of sets $C, D \subseteq X$

⁴Also called “ergodic” in [7].

such that $\mu(C \cap T^g D) > 0$ for some $g \in G$, there exists $h \in S$ such that $\mu(C \cap T^h D) > 0$.

- *density expanding* if for every set $A \subseteq G$ having $d^*(A) > 0$, $d^*(S + A) = 1$.
- *expanding for minimal G -systems* if for every minimal topological G -system (X, T) there is a dense G_δ set $Y \subseteq X$, such that $\{T^g y : g \in S\}$ is dense in X for all $y \in Y$.
- *transitive for minimal G -systems* if for every minimal topological G -system and every pair of nonempty open sets $U, V \subseteq X$ there exists $h \in S$ such that $U \cap T^h V \neq \emptyset$.
- *chromatically expanding* if for every piecewise syndetic set $A \subseteq G$, $d^*(S + A) = 1$.

We have the following equivalences and implications. The equivalences are proved in Lemma 5.3.

$$\begin{aligned}
 (R_4^\bullet) &\iff S \text{ is measure expanding} \\
 &\iff S \text{ is measure transitive} \\
 &\implies S \text{ is density expanding,} \\
 (R_5^\bullet) &\iff S \text{ is expanding for minimal } G\text{-systems} \\
 &\iff S \text{ is transitive for minimal } G\text{-systems} \\
 &\implies S \text{ is chromatically expanding,}
 \end{aligned}$$

We now discuss the implications “ $(R_4^\bullet) \implies S$ is density expanding” and “ $(R_5^\bullet) \implies S$ is chromatically expanding.”

It is well known that if S is measure expanding, then S is density expanding – see Correspondence Principle II in [7] or Proposition 2.3 of [2]. A counterexample to implication (3) of Question 1.5 for a given G would therefore show that $(R_6^\bullet) \not\implies (R_4)$. The implication “ $(R_5^\bullet) \implies S$ is chromatically expanding” has a proof analogous to the proof of “ $(R_4^\bullet) \implies S$ is density expanding.” Whether density expanding implies measure expanding for $G = \mathbb{Z}$ is asked in [20] and remains open. In general we have the following question which is open for every countable abelian group G .

Question 2.15. Let S be a subset of a countable abelian group G .

- Does S being density expanding imply (R_4^\bullet) ?
- Does S being chromatically expanding imply (R_5^\bullet) ?

If every density expanding S satisfies (R_4^\bullet) , then Part (1) of Question 1.5 is equivalent to Part (ii) of Question 2.2. We also have the following

variant of Question 1.5, both parts of which are open for every G besides G_2 .

Question 2.16. Let $S \subseteq G$.

- (i) Does (R_5^\bullet) imply that S is density expanding?
- (ii) Does (R_5^\bullet) imply that $S + A$ is piecewise syndetic whenever $d^*(A) > 0$?

For $G = G_2$, Theorem 1.6 answers both parts of Question 2.16 in the negative. A positive solution to Problem 4.4 together with the results of [23] would answer both parts of Question 2.16 in the negative for $G = G_p$, where p is any prime.

3. VECTOR SPACES OVER \mathbb{F}_p

In this section we state some definitions and conventions needed for the proof of Theorems 1.1, 1.2, 1.4, and 1.6. We identify a useful presentation of $G_p := \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ and some associated subgroups. We use an arbitrary prime p so that we can formulate Problem 4.4, but we specialize to the case $p = 2$ in our proofs. This material is also presented in Section 2 of [23].

If p is a prime number, let $\mathbb{F}_p (= \mathbb{Z}/p\mathbb{Z})$ denote the finite field with p elements. We write the elements of \mathbb{F}_p as $0, 1, \dots, p-1$. Consider the countable direct sum $G_p := \bigoplus_{n=1}^{\infty} \mathbb{F}_p$.

3.1. Presentation of G_p . Let $\Omega := \{0, 1\}^{\mathbb{N}}$, and write elements of Ω as $\omega = \omega_1\omega_2\omega_3\dots$. For each $n \in \mathbb{N}$ let $\Omega_n = \{0, 1\}^{\{1, \dots, n\}}$ and $\pi_n : \Omega \rightarrow \Omega_n$ be the projection map given by $\pi_n(\omega) = \omega_1 \dots \omega_n$. Let Γ_p be the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ with the group operation of pointwise addition.

For each $n \in \mathbb{N}$, let $G_p^{(n)}$ be the subgroup of Γ_p consisting of functions of the form $f \circ \pi_n$, where $f : \Omega_n \rightarrow \mathbb{F}_p$. Observing that $G_p^{(n)} \subseteq G_p^{(n+1)}$ for each n , we let $\tilde{G}_p := \bigcup_{n \in \mathbb{N}} G_p^{(n)}$. Then \tilde{G}_p is a countable abelian group isomorphic⁵ to G_p . Our constructions are easier to work with from the perspective of \tilde{G}_p rather than the standard presentation of a countable direct sum, so from now on we let G_p denote the group \tilde{G}_p .

We observe that $G_p^{(n)}$ is isomorphic to $(\mathbb{F}_p)^{\Omega_n}$, and we will identify elements of $G_p^{(n)}$ with elements of $(\mathbb{F}_p)^{\Omega_n}$. The identification is given by $g \leftrightarrow \tilde{g}$ if and only if $g = \tilde{g} \circ \pi_n$ for $\tilde{g} \in (\mathbb{F}_p)^{\Omega_n}$.

⁵One can construct the isomorphism by hand, but when p is prime it suffices to observe that both G_p and \tilde{G}_p are countably infinite vector spaces over the finite field \mathbb{F}_p .

Let $G_p^{(0)}$ denote the group of constant functions $f : \Omega \rightarrow \mathbb{F}_p$, so that $G_p^{(0)} \subseteq G_p^{(1)}$. Let $\mathbf{1} \in G_p$ denote the constant function where $\mathbf{1}(\omega) = 1 \in \mathbb{F}_p$ for every $\omega \in \Omega$. For $x \in \mathbb{F}_p$, define $x\mathbf{1}$ to be the constant function having $x\mathbf{1}(\omega) = x$ for all $\omega \in \Omega$, and let $\mathbf{0}$ denote $0\mathbf{1}$, the identity element of G_p .

Remark 3.1. G_p is the group of continuous functions $g : \Omega \rightarrow \mathbb{F}_p$, where Ω has the product topology and \mathbb{F}_p has the discrete topology. Alternatively, G_p is the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ where $g(\omega)$ depends on only finitely many coordinates of ω .

3.2. Cylinder sets. If $\tau \in \Omega_n$, let $[\tau] \subseteq \Omega$ be $\pi_n^{-1}(\tau)$, so that $[\tau] := \{\omega \in \Omega : \omega_i = \tau_i \text{ for } 1 \leq i \leq n\}$. We call $[\tau]$ a *cylinder set*. Each cylinder set $[\tau]$ is homeomorphic to Ω by the map $\theta : [\tau] \rightarrow \Omega$, $\theta(\omega) := \omega_{n+1}\omega_{n+2}\dots$.

Observe that $G_p^{(n)}$ is the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ which are constant on the cylinder sets $[\tau]$, where $\tau \in \Omega_n$.

Definition 3.2. If $E \subseteq \Omega$, let $|E|_n := |\{\tau \in \Omega_n : [\tau] \subseteq E\}|$.

The above definition will usually be applied to sets of the form $g^{-1}(S)$ where $g \in G_p^{(n)}$ and $S \subseteq \mathbb{F}_p$. We list some relevant properties.

Observation 3.3. (C1) $|E|_n \leq 2^n$ for all $E \subseteq \Omega$.

(C2) For an element $g \in G_p^{(n)}$, if $g = \tilde{g} \circ \pi_n$, then $|g^{-1}(1)|_n = |\tilde{g}^{-1}(1)|$.

(C3) If $g \in G_p^{(n)}$, $A, B \subseteq \mathbb{F}_p$, and $A \cap B = \emptyset$, then

$$|g^{-1}(A) \cup g^{-1}(B)|_n = |g^{-1}(A)|_n + |g^{-1}(B)|_n.$$

3.3. Restrictions to cylinders. Given $m, n \in \mathbb{N}$ with $m < n$, a string $\tau \in \Omega_m$, and an element $g \in G_p^{(n)}$, we define $g|_\tau$ to be the element of $G_p^{(n-m)}$ satisfying $g|_\tau(\omega) = g(\tau\omega)$ for all $\omega \in \{0, 1\}^{(n-m)}$, where $\tau\omega \in \Omega$ is the concatenation of τ and ω . To give an explicit example: for $n = 5$, $m = 2$, and $g : \Omega \rightarrow \mathbb{F}_7$, if $\tau = 01$ and $\omega \in \Omega$, then $g|_\tau(\omega) = g(01\omega_1\omega_2\omega_3\dots)$. With this notation we can identify $g \in G_p^{(n)}$ with the function $f : \Omega_m \rightarrow G_p^{(n-m)}$, where $f(\tau) := g|_\tau$ for each $\tau \in \Omega_m$. This identification is used in Definition 4.10, where the sets A of Theorems 1.2 and 1.6 are defined.

3.4. Upper Banach density in G_p . Since $G_p = \bigcup_{n=1}^{\infty} G_p^{(n)}$, the sequence $(G_p^{(n)})_{n \in \mathbb{N}}$ is a Følner sequence for G_p , as mentioned in Section 2.7. Consequently, we have $d^*(A) \geq \limsup_{n \rightarrow \infty} \frac{|A \cap G_p^{(n)}|}{|G_p^{(n)}|}$ for every $A \subseteq G_p$.

3.5. Hamming Balls. For $n, k \in \mathbb{N} \cup \{0\}$, let $U(n, k)$ be the set of $g \in G_p^{(n)}$ satisfying $|\{\omega \in \Omega : g(\omega) \neq 0\}|_n \leq k$ (cf. Definition 3.2). This is the *Hamming ball of scale n and radius k around $\mathbf{0} \in G_p^{(n)}$* . In other words, $U(n, k)$ is the set of $g \in G_p$ which are constant on the cylinder sets $[\tau]$ for $\tau \in \Omega_n$ and $g|_\tau = \mathbf{0}$ for at least $|\Omega_n| - k$ such τ .

For $g \in G_p$, let $V(n, k) := U(n, k) + \mathbf{1}$, so that

$$V(n, k) = \{g \in G_p^{(n)} : |\{\omega \in \Omega : g(\omega) \neq 1\}|_n \leq k\}.$$

We call $V(n, k)$ the *Hamming ball of scale n and radius k around $\mathbf{1}$* .

Remark 3.4. We call the sets $U(n, k)$ and $V(n, k)$ “Hamming balls” as we may identify elements of $G_p^{(n)}$ with strings of length 2^n from the alphabet \mathbb{F}_p . With this identification $U(n, k)$ is the set of strings differing from the constant 0 string in at most k coordinates.

Definition 3.5. Let $k, n \in \mathbb{N}$, $\delta > 0$. A set $S \subseteq G_p^{(n)}$ is a

- *set of δ -density recurrence* if for every $A \subseteq G_p^{(n)}$ having $|A| \geq \delta |G_p^{(n)}|$, $(A - A) \cap S \neq \emptyset$.
- *set of k -chromatic recurrence* if for all partitions $G_p^{(n)} = \bigcup_{j=1}^k A_j$, there is a j such that $(A_j - A_j) \cap S \neq \emptyset$.

In $G_2^{(n)}$, the sets $V(n, k)$ have the following important properties.

- (V1) For $\delta < 1/2$ and n much larger than k , $V(n, k)$ is not a set of δ -density recurrence: when n is very large compared to k , there are sets $A \subseteq G_2^{(n)}$ having $|A| > |G_2^{(n)}|(\frac{1}{2} - \varepsilon)$, such that $(A - A) \cap V(n, k) = \emptyset$. In fact

$$A := \{g \in G_2^{(n)} : |g^{-1}(1)|_n > \frac{1}{2}|\Omega_n| + k\}$$

is such a set.

- (V2) $V(n, k)$ is a set of k -chromatic recurrence: if $G_2^{(n)} = \bigcup_{j=1}^k A_j$, there is a j such that $(A_j - A_j) \cap V(n, k) \neq \emptyset$.
- (V3) $V(n, k)$ is a set of density recurrence when k is comparable to a fixed multiple of n : for fixed $c, \delta > 0$, there exists N such that $(A - A) \cap V(n, \lfloor cn \rfloor) \neq \emptyset$ whenever $n \geq N$ and $A \subseteq G_2^{(n)}$ has $|A| \geq \delta |G_2^{(n)}|$.

Property (V2) is established in the proof of Lemma 4.1. Property (V1) is proved in Lemmas 4.8 and 4.14. Property (V3), which we do not use in this paper, is a corollary of a theorem of Kleitman [29], as shown in [13].

Each of the constructions in [30], [32, Theorem 1.2], and [33, Theorem 3.35] prove that $(R_5) \not\Rightarrow (R_4)$ by finding sets $\tilde{V}(n, k) \subseteq \mathbb{Z}$ imitating $V(n, k)$ and exploiting Properties (V1) and (V2), taking a union $\bigcup_{i=1}^{\infty} \tilde{V}(n_i, k_i)$ to construct the desired example. Similarly, the constructions of [13], [14], (proving $(R_4) \not\Rightarrow (R_3)$), and [21] (proving $(R_4^\bullet) \not\Rightarrow (R_3)$) find sets $\tilde{V}(n, k) \subseteq \mathbb{Z}$ imitating $V(n, k)$ and exploiting Property (V3), taking a union of these to get the desired example. Every known example distinguishing some pair of the properties $(R_3), (R_4), (R_5), (R_6)$, or $(R_3^\bullet), (R_4^\bullet), (R_5^\bullet), (R_6^\bullet)$ follows this rough outline. It would be interesting to find, for example, a set $S \subseteq \mathbb{Z}$ satisfying (R_5) but not (R_4) , which is not constructed in this way. To be more specific, we pose the following question.

Question 3.6. (i) Is there a set $S \subseteq S_3 := \{n^3 : n \in \mathbb{N}\}$ satisfying (R_5) but not (R_4) ?
(ii) Is there a set $S \subseteq S_{5/2} := \{\lfloor n^{5/2} \rfloor : n \in \mathbb{N}\}$ satisfying (R_5^\bullet) but not (R_4) ?

We use S_3 and $S_{5/2}$ because it appears that the constructions of [30] and [32, 33] cannot produce subsets of S_3 and $S_{5/2}$ which are sets of topological recurrence, so a significant modification of the technique seems to be necessary to provide an affirmative answer to either part of Question 3.6. On the other hand, it would be interesting to find a set of measurable recurrence with the property that every subset thereof which is a set of topological recurrence is also a set of measurable recurrence, so we formulate the following question, which is open for every group G .

Question 3.7. Fix a countable abelian group G . Is there a set of measurable recurrence $S \subseteq G$ such that every subset of S which is a set of topological recurrence is also a set of measurable recurrence?

A negative answer to either part of Question 3.6 would provide a positive answer to Question 3.7, as it is well known that S_3 is a set of measurable recurrence (by a theorem due to Furstenberg [15] and Sárközy [38], independently) and $S_{5/2}$ satisfies (R_1) (see [9], for example).

4. PROOF OF THEOREM 1.2

In this section we construct the sets $S, A \subseteq G_2$ described in Theorem 1.2. We maintain the notation and conventions of Section 3. The set S will be a union of some of the $V(n, k)$ (defined in Section 3.5), so

we begin by showing that the $V(n, k)$ satisfy a quantitative version of chromatic recurrence.

4.1. Chromatic recurrence properties of the $V(n, k)$.

Lemma 4.1. *Let $(n_i)_{i \in \mathbb{N}}, (k_i)_{i \in \mathbb{N}}$ be increasing sequences of natural numbers and let $(g_i)_{i \in \mathbb{N}}$ be a sequence of elements of G_2 with $g_i \in G_2^{(n_i)}$ for each i . Then $\bigcup_{i \in \mathbb{N}} (g_i + U(n_i, k_i)) \setminus \{0\}$ is a set of chromatic recurrence, and therefore a set of topological recurrence.*

Consequently, every translate of $S := \bigcup_{i=1}^{\infty} V(n_i, k_i)$ is a set of topological recurrence in G_2 , and in fact, for every $g \in G_2$, $(g + S) \setminus \{0\}$ is a set of topological recurrence.

As in [14, 30, 32, 33], we prove Lemma 4.1 as a consequence of the following theorem of Lovász [31].

Theorem 4.2. *Let $k, r \in \mathbb{N}$, and let E be the set of r -element subsets of $\{1, \dots, 2r + k\}$. If $E = \bigcup_{j=1}^k E_j$, there is a $j \leq k$ and a disjoint pair of elements $e_1, e_2 \in E$ such that $e_1, e_2 \in E_j$.*

We also need the following elementary lemma, which we prove as a very special case of the Poincaré Recurrence Theorem.

Lemma 4.3. *For $k < n \in \mathbb{N}$, if $G_2^{(n)} = \bigcup_{j=1}^k A_j$, then*

$$(A_j - A_j) \cap U(n, 2k + 2) \notin \{\emptyset, \{0\}\}$$

for some $j \leq k$.

Proof. For some j we have $|A_j| \geq \frac{1}{k}|G_2^{(n)}|$; fix such a j . Note that $U(n, k + 1), U(n, 2k + 2) \subseteq G_2^{(n)}$, $U(n, 2k + 2)$ contains the difference set $U(n, k + 1) - U(n, k + 1)$, and $|U(n, k + 1)| \geq k + 1$. The sets $A_j + u$, $u \in U(n, k + 1)$ cannot all be mutually disjoint, since that would imply

$$|G_2^{(n)}| \geq |U(n, k + 1)| \cdot |A_j| \geq \frac{k+1}{k}|G_2^{(n)}| > |G_2^{(n)}|.$$

Hence there exist $u_1 \neq u_2 \in U(n, k + 1)$ such that $(A_j + u_1) \cap (A_j + u_2) \neq \emptyset$, meaning there exist $a, b \in A_j$ such that $a - b = u_1 - u_2$. Since $u_1 \neq u_2$ and $u_1 - u_2 \in U(n, 2k + 2)$, we have shown that $(A_j - A_j) \cap U(n, 2k + 2) \notin \{\emptyset, \{0\}\}$. \square

Proof of Lemma 4.1. We will prove the following.

Claim. *Fix $k < n \in \mathbb{N}$. If $g \in G_2^{(n)}$ and $G_2^{(n)} = \bigcup_{j=1}^k A_j$, then for some $j \leq k$ we have $(A_j - A_j) \cap (g + U(n, 3k + 3)) \notin \{\emptyset, \{0\}\}$.*

Proof of Claim. Fix $g \in G_2^{(n)}$, $k \in \mathbb{N}$, and a partition $G_2^{(n)} = \bigcup_{j=1}^k A_j$. Let $X_1 = g^{-1}(1)$, so that $X_1 \subseteq \pi_n^{-1}(\Omega_n) \subset \Omega$. We consider two cases based on $|X_1|_n$ (defined in Section 3.3).

Case 1. $|X_1|_n \geq 2 + k$. In this case write $|X_1|_n = 2r + k$ if $|X_1|_n - k$ is even and write $|X_1|_n = 2r + k + 1$ if $|X_1|_n - k$ is odd, where $r \in \mathbb{N}$. Let

$$D := \{h \in G_2^{(n)} : h^{-1}(1) \subseteq X_1 \text{ and } |h^{-1}(1)|_n = r\}$$

Identify D with the r -element subsets of $\{\tau \in \Omega_n : [\tau] \subset X_1\}$, where the identification is given by $h \leftrightarrow \{\tau \in \Omega_n : h([\tau]) = \{1\}\}$. For $1 \leq j \leq k$, let $D_j = A_j \cap D$, so that $D = \bigcup_{j=1}^k D_j$. Theorem 4.2 implies that some D_j contains two disjoint elements of D , say h_1 and h_2 . The difference $h_1 - h_2$ satisfies $(h_1 - h_2)(\omega) = 0$ for $\omega \notin X_1$, and $|(h_1 - h_2)^{-1}(1)|_n = 2r$, meaning $(h_1 - h_2)([\tau]) \neq g([\tau])$ for at most $k + 1$ values of $\tau \in \Omega_n$. It follows that

$$h_1 - h_2 \in (g + U(n, k + 1)) \subseteq (g + U(n, 3k + 3)),$$

and $h_1 - h_2 \neq 0$. Since $h_1, h_2 \in D_j \subseteq A_j$, this concludes the proof of the Claim in Case 1.

Case 2. $|X_1|_n < 2 + k$. In this case $g + U(n, 3k + 3)$ contains the Hamming ball $U(n, 2k + 2)$, which has the desired property, by Lemma 4.3. This completes the proof of the Claim. \square

To complete the proof of Lemma 4.1, let $(n_i)_{i \in \mathbb{N}}$, $(k_i)_{i \in \mathbb{N}}$, and $(g_i)_{i \in \mathbb{N}}$ be as in the statement of the lemma. Let $R := \bigcup_{i=1}^{\infty} g_i + U(n_i, k_i)$, $k \in \mathbb{N}$, and let $G_2 = \bigcup_{j=1}^k A_j$ be a partition of G_2 . Fix $i \in \mathbb{N}$ so that $n_i, k_i \geq 3k + 3$. Let $A'_j = A_j \cap G_2^{(n_i)}$ for each $j \leq k$. By the Claim, there exists $j \leq k$ such that $(A'_j - A'_j) \cap (g_i + U(n_i, k_i)) \notin \{\emptyset, \{0\}\}$. Consequently $(A_j - A_j) \cap R \notin \{\emptyset, \{0\}\}$. Since the partition of G_2 was arbitrary, we have shown that $R \setminus \{0\}$ is a set of chromatic recurrence, as desired. Proposition 2.9 then implies $R \setminus \{0\}$ is a set of topological recurrence.

To prove that every translate of $S := \bigcup_{i=1}^{\infty} V(n_i, k_i)$ is a set of topological recurrence, let $g \in G_2$ and choose j sufficiently large that $g \in G_2^{(n_j)}$. Then $g + S \supseteq \bigcup_{i=j}^{\infty} g + \mathbf{1} + U(n_i, k_i)$, which is a set of topological recurrence, by the preceding paragraph. \square

Problem 4.4. Let p be an odd prime and let $V(n, k) \subseteq G_p$ be as defined in Section 3.5. Prove or disprove:

- (*) If $n_i, k_i \rightarrow \infty$, then every translate of $S := \bigcup_{i=1}^{\infty} V(n_i, k_i)$ is a set of topological recurrence.

If the statement $(*)$ is true for a given p , then the results of [23] imply $(R_5^\bullet) \not\Rightarrow (R_4)$ for G_p , and that there are sets $S, A \subseteq G_p$ having $d^*(A) > 0$ and every translate of S is a set of topological recurrence, while $S + A$ is not piecewise syndetic. This would provide a negative answer to Part (iii) of Question 2.2 and both parts of Question 2.16 for $G = G_p$. If the statement $(*)$ is false for some odd prime p , the results of [23] provide an example showing that $(R_6^\bullet) \not\Rightarrow (R_5)$ for the corresponding G_p , giving a negative answer to Part (i) of Question 2.2.

4.2. Some dense subsets of G_2 . Our second task is to construct the sets A of Theorems 1.2 and 1.6. We will find, for each $\varepsilon > 0$, a set A such that $d^*(A) > \frac{1}{2} - \varepsilon$, while $(A - A) \cap S = \emptyset$ for some $S = \bigcup_{j=1}^{\infty} V(n_j, k_j)$ where $n_j \rightarrow \infty$, $k_j \rightarrow \infty$. Exhibiting such A and S will prove Theorem 1.2, as Lemma 4.1 implies $(B - B + g) \cap S \neq \emptyset$ whenever B is piecewise syndetic and $g \in G_2$. We will also show that $(A' - A') \cap S = \emptyset$, where $A' = A + S$, leading to a proof of Theorem 1.6.

The following definition uses the notation $|\cdot|_n$ defined in Section 3.2.

Definition 4.5. For $n, m \in \mathbb{N}$ and $i \in \mathbb{Z}/2\mathbb{Z}$, let

$$A_i(n, m) := \{g \in G_2^{(n)} : |g^{-1}(i)|_n > \frac{1}{2}|\Omega_n| + m\}.$$

So $A_i(n, m)$ is the set of functions $g : \Omega \rightarrow \mathbb{Z}/2\mathbb{Z}$ which are constant on the cylinder sets $[\tau]$ for $\tau \in \Omega_n$, and the number of $\tau \in \Omega_n$ such that $g([\tau]) = \{i\}$ is greater than $\frac{1}{2}|\Omega_n| + m$.

Remark 4.6. The $A_i(n, m)$ are examples of niveau sets, first explicitly used in additive combinatorics by Ruzsa in [35, 36]; see [40] for exposition and an application.

While the $A_0(n, m)$ are essentially the Hamming balls $U(n, k)$, where $k = \frac{1}{2}|\Omega_n| - m$, we do not treat them as such. Instead we view the $A_i(n, m)$ as the base case of the inductive Definition 4.10.

Remark 4.7. Letting $Z(n, m) = G_2^{(n)} \setminus (A_0(n, m) \cup A_1(n, m))$, we have a partition of $G_2^{(n)}$ into three sets (it is easy to check that $A_0(n, m)$ and $A_1(n, m)$ are disjoint). When n is very large compared to m , $|A_i(n, m)|$ is close to $\frac{1}{2}|G_2^{(n)}|$, while $(A_i(n, m) - A_i(n, m)) \cap V(n, m) = \emptyset$, as we shall prove. In light of these facts and Lemma 4.1 we have a natural candidate for the set A in Theorem 1.2, namely $\bigcup_{j=1}^{\infty} A_1(n_j, m_j)$, where the $n_j \rightarrow \infty$ rapidly and the $m_j \rightarrow \infty$ slowly. But this choice of A will not work: setting

$$A' := A_1(n_1, m_1) \cup A_2(n_2, m_2), \quad S' := V(n_1, m_1) \cup V(n_2, m_2),$$

the desired disjointness $(A' - A') \cap S' = \emptyset$ is not true in general. Instead of sets such as A' , we use sets which are easily understood in terms of their translates by elements of $V(n_1, m_1) \cup V(n_2, m_2)$. Considering elements $g \in G_2^{(n_2)}$ as functions $g : \Omega_{n_1} \rightarrow G_2^{(n_2-n_1)}$, we think of $A_i(n_2 - n_1, m_2)$ as playing the role of $i \in \mathbb{Z}/2\mathbb{Z}$, and we let $A_i((n_1, m_1), (n_2, m_2))$ be the set of $g \in G_2^{(n_2)}$ satisfying $g|_\tau \in A_i(n_2 - n_1, m_2)$ for greater than m_1 values of $\tau \in \Omega_{n_1}$. As we shall see in Lemma 4.12, it is easy to understand $g+v$ for $g \in A_i((n_1, m_1), (n_2, m_2))$ and $v \in V(n_1, m_1) \cup V(n_2, m_2)$. If $n_2 - n_1$ is very large compared to m_2 , then $|A_i(n_2 - n_1, m_2)|$ is very close to $\frac{1}{2}|G_2^{(n_2-n_1)}|$, so estimating $|A_i((n_1, m_1), (n_2, m_2))|$ is not difficult. Furthermore, this construction can be iterated to produce sets $A \subseteq G_2^{(n_i)}$ whose translates by elements of $V(n_1, m_1) \cup \dots \cup V(n_l, m_l)$ are easily understood.

The following lemma summarizes the relevant properties of the sets $A_i(n, m)$. Recall from Section 3.1 that $\mathbf{1}$ denotes the element $g \in G_2$ having $g(\omega) = 1 \in \mathbb{Z}/2\mathbb{Z}$ for all $\omega \in \Omega$.

Lemma 4.8. *For all $n, m, \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $k < m$, and $i \in \mathbb{Z}/2\mathbb{Z}$, we have*

- (i) $A_i(n, m) + \mathbf{1} = A_{i+1}(n, m)$,
- (ii) $A_i(n, m) + U(n, k) \subseteq A_i(n, m - k)$,
- (iii) $A_i(n, m) + U(n, k) + \mathbf{1} \subseteq A_{i+1}(n, m - k)$,
- (iv) $A_i(n, m) \cap A_{i+1}(n, m - k) = \emptyset$,
- (v) *If $m' < m$, then $A_i(n, m) \subseteq A_i(n, m')$.*

Proof. Part (i) follows from the definition of $A_i(n, m)$ and the fact that $(g + \mathbf{1})^{-1}(i) = g^{-1}(i + 1)$ for all $g \in G_2$.

To prove Part (ii), note that for each $g \in G_2^{(n)}$, $u \in U(n, k)$, we have $g([\tau]) = (g + u)([\tau])$ for at least $|\Omega_n| - k$ values of $\tau \in \Omega_n$. We then have $|(g + u)^{-1}(i)|_n \geq |g^{-1}(i)|_n - k$ for every such g , and Part (ii) now follows from the definition of $A(n, m)$.

Part (iii) follows directly from parts (i) and (ii).

To prove Part (iv) we exploit Observation 3.3. Assume, to get a contradiction, that $A_i(n, m) \cap A_{i+1}(n, m - k) \neq \emptyset$, and let g be an element of the intersection. We then have $|g^{-1}(i)|_n \geq \frac{1}{2}|\Omega_n| + m$ and $|g^{-1}(i + 1)|_n \geq \frac{1}{2}|\Omega_n| + m - k$. Since $g^{-1}(i) \cap g^{-1}(i + 1) = \emptyset$, we then have $|\Omega_n| \geq |\Omega_n| + 2m - k$, a contradiction.

Part (v) follows immediately from the definition of $A_i(n, m)$. \square

Remark 4.9. The $A_i(n, m)$ are Hamming balls of radius $\frac{1}{2}|\Omega_n| - m - 1$ around $i\mathbf{1} \in G_2^{(n)}$, so Parts (ii) and (iii) of Lemma 4.8 are consequences of the fact that $U(n, k) + U(n, k') = U(n, k + k')$.

In the following definition we will use the restrictions $g|_\tau$, defined in Section 3.3.

Definition 4.10. For all $l \in \mathbb{N}$, $i \in \mathbb{Z}/2\mathbb{Z}$, and sequences of l pairs of natural numbers $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$ where $n_1 < \dots < n_l$, we will define a set $A_i(\mathbf{n}_l) \subseteq G_2^{(n_l)}$. For $l = 1$, $A_i(n_1, m_1)$ is defined in Definition 4.5. For $l \geq 2$, we define $A_i(\mathbf{n}_l)$ inductively, assuming $A_i((n'_1, m'_1), \dots, (n'_{l-1}, m'_{l-1}))$ is defined for every sequence of pairs where $n'_1 < \dots < n'_{l-1}$. In particular, $A_i(\bar{\mathbf{n}}_{l-1})$ is defined, where $\bar{\mathbf{n}}_{l-1} := (n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)$. For each $i \in \mathbb{Z}/2\mathbb{Z}$, define $A_i((n_1, m_1), \dots, (n_l, m_l))$ to be the set of $g \in G_2^{(n_l)}$ such that

$$(4.1) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in A_i(\bar{\mathbf{n}}_{l-1})\}| > \frac{1}{2}|\Omega_{n_1}| + m_1.$$

For example, $A_1((10, 3), (50, 7))$ is the set of $g \in G_2^{(50)}$ such that $g|_\tau \in A_1(40, 7)$ for $> \frac{1}{2}|\Omega_{10}| + 3$ values of $\tau \in \Omega_{10}$.

Remark 4.11. The proofs of Theorems 1.2 and 1.6 are a straightforward exploitation of Definition 4.10. The remaining proofs in this subsection are tedious due to the inductive nature of the definition.

We adopt the following conventions in the sequel.

- The symbol \mathbf{n}_l abbreviates the symbol $(n_1, m_1), \dots, (n_l, m_l)$.
- The symbol $\bar{\mathbf{n}}_{l-1}$ abbreviates $(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)$.

Lemma 4.12. For all $l \in \mathbb{N}$, sequences $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$ with $n_j, m_j \in \mathbb{N}$, $n_1 < \dots < n_l$, $i \in \mathbb{Z}/2\mathbb{Z}$, $j \in \{1, \dots, l\}$, $k_j \in \mathbb{N} \cup \{0\}$, $k_j < m_j$ we have

$$(i) \quad A_i(\mathbf{n}_l) + \mathbf{1} = A_{i+1}(\mathbf{n}_l),$$

$$(ii) \quad \text{if } u \in U(n_j, k_j), \text{ then}$$

$$A_i(\mathbf{n}_l) + u \subseteq A_i((n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)),$$

$$(iii) \quad \text{if } u \in U(n_j, k_j), \text{ then}$$

$$A_i(\mathbf{n}_l) + u + \mathbf{1} \subseteq A_{i+1}((n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)),$$

$$(iv) \quad \text{the sets } A_i(\mathbf{n}_l) \text{ and}$$

$$A'_{i+1} := A_{i+1}((n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l))$$

are disjoint. In particular $A_0(\mathbf{n}_l) \cap A_1(\mathbf{n}_l) = \emptyset$.

(v) If $m'_j \leq m_j$ then

$$A_i(\mathbf{n}_l) \subseteq A_i((n_1, m_1), \dots, (n_j, m'_j), \dots, (n_l, m_l)).$$

Proof. We prove each of these statements by induction on l . For each of Parts (i)-(v), the base case of the induction is $l = 1$, which is the corresponding part of Lemma 4.8. We now fix $l \in \mathbb{N}$, $l > 1$. For the induction hypothesis, we assume each of (i) - (v) holds for all sequences $(n'_1, m'_1), \dots, (n'_{l-1}, m'_{l-1})$, where $n'_j, m'_j \in \mathbb{N}$ and $n'_1 < \dots < n'_{l-1}$. In particular, given a sequence $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$ of length l , each of (i)-(v) holds for the sequence $\bar{\mathbf{n}}_{l-1} = (n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)$ of length $l - 1$.

To prove (i), let $g \in A_i(\mathbf{n}_l)$, so that $g|_\tau \in A_i(\bar{\mathbf{n}}_{l-1})$ for greater than $\frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$. For each such τ , $(g + \mathbf{1})|_\tau \in A_{i+1}(\bar{\mathbf{n}}_{l-1})$, by the induction hypothesis. Then $(g + \mathbf{1})|_\tau \in A_{i+1}(\bar{\mathbf{n}}_{l-1})$ for greater than $\frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$, so $g + \mathbf{1} \in A_{i+1}(\mathbf{n}_l)$ by definition. By symmetry we conclude that if $g \in A_{i+1}(\mathbf{n}_l)$, then $g - \mathbf{1} \in A_i(\mathbf{n}_l)$, and we conclude that $A_i(\mathbf{n}_l) + \mathbf{1} = A_{i+1}(\mathbf{n}_l)$.

To prove (ii) we consider two cases.

Case 1: $j = 1$. Let $g \in A_i(\mathbf{n}_l)$ and $u \in U(n_1, k_1)$. Then $g|_\tau \in A_i(\bar{\mathbf{n}}_{l-1})$ for greater than m_1 values of $\tau \in \Omega_{n_1}$, while $(g + u)|_\tau = g|_\tau$ for all but k_1 values of $\tau \in \Omega_{n_1}$. It follows that

$$|\{\tau \in \Omega_{n_1} : (g + u)|_\tau \in A_i(\bar{\mathbf{n}}_{l-1})\}| > m_1 - k_1,$$

so $g + u \in A_i((n_1, m_1 - k_1), \dots, (n_l, m_l))$, by definition.

Case 2: $2 \leq j \leq l$. Again let $g \in A_i(\mathbf{n}_l)$ and $u \in U(n_j, k_j)$. We have $u|_\tau \in U(n_j - n_1, k_j)$ for each $\tau \in \Omega_{n_1}$. Then for each such τ where $g|_\tau \in A_i(\bar{\mathbf{n}}_{l-1})$, the induction hypothesis implies

$$(g + u)|_\tau \in A_i((n_2 - n_1, m_2), \dots, (n_j - n_1, m_j - k_j), \dots, (n_l - n_1, m_l)).$$

The above inclusion then occurs for $> \frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$, so $g + u \in A_i((n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l))$. This completes the proof of Part (ii).

Part (iii) follows immediately from Parts (i) and (ii).

We prove Part (iv) by induction on l . The base case of the induction is $l = 1$, which is Part (iv) of Lemma 4.8. For the induction step we set $i = 1$, as the case $i = 0$ follows by symmetry. Let $l > 1$ and assume, to get a contradiction, that $A_1(\mathbf{n}_l)$ and A'_0 are not disjoint, and let $g \in A_1(\mathbf{n}_l) \cap A'_0$. We now consider two cases.

Case 1: $j = 1$. The induction hypothesis implies $A_0(\bar{\mathbf{n}}_{l-1})$ and $A_1(\bar{\mathbf{n}}_{l-1})$ are disjoint. We have

$$(4.2) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in A_1(\bar{\mathbf{n}}_{l-1})\}| > \frac{1}{2}|\Omega_{n_1}| + m_1, \text{ since } g \in A_1(\mathbf{n}_l),$$

$$(4.3) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in A_0(\bar{\mathbf{n}}_{l-1})\}| > \frac{1}{2}|\Omega_{n_1}| + m_1 - k_1, \text{ since } g \in A'_0.$$

Inequalities (4.2) and (4.3) together imply $|\Omega_{n_1}| > |\Omega_{n_1}| + 2m_1 - k_1$, contradicting the assumption $k_1 \leq m_1$.

Case 2: $j > 1$. The induction hypothesis implies that the sets $A_1(\bar{\mathbf{n}}_{l-1})$ and

$$A''_0 := A_0((n_2 - n_1, m_2), \dots, (n_j - n_1, m_j - k_j), \dots, (n_l - n_1, m_l))$$

are disjoint. Then

$$(4.4) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in A_1(\bar{\mathbf{n}}_{l-1})\}| > \frac{1}{2}|\Omega_{n_1}| + m_1, \text{ since } g \in A_1(\mathbf{n}_l),$$

$$(4.5) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in A''_0\}| > \frac{1}{2}|\Omega_{n_1}| + m_1, \text{ since } g \in A'_0.$$

Inequalities (4.4) and (4.5) together imply $|\Omega_{n_1}| > |\Omega_{n_1}| + 2m_1$, a contradiction. This completes the proof of Part (iv).

For Part (v) we again use induction on l . The base case, $l = 1$, is Part (v) of Lemma 4.8, so we establish the induction step. Assume $l > 1$. The induction hypothesis implies

$$A_i(\bar{\mathbf{n}}_{l-1}) \subseteq A_i((n_2 - n_1, m_2), \dots, (n_j - n_1, m'_j), \dots, (n_l - n_1, m_l)),$$

and the definition of $A_i(\mathbf{n}_l)$ then implies the conclusion. This completes the proof of Part (v) and the proof of the Lemma. \square

Lemma 4.13. *For all $i \in \mathbb{Z}/2\mathbb{Z}$ and all $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$, $\mathbf{n}_{l+1} = (n_1, m_1), \dots, (n_{l+1}, m_{l+1})$ where $n_1 < \dots < n_{l+1}$, $n_j, m_j \in \mathbb{N}$, $A_i(\mathbf{n}_l) \subseteq A_i(\mathbf{n}_{l+1})$.*

Proof. We consider the case $i = 1$. The case $i = 0$ follows by symmetry.

We proceed by induction on l . The base case is the containment $A_1(n_1, m_1) \subseteq A_1((n_1, m_1), (n_2, m_2))$. Let $g \in A_1(n_1, m_1)$. We must show that $g|_\tau \in A_1(n_2 - n_1, m_2)$ for $> m_1$ values of $\tau \in \Omega_{n_1}$. In fact $g|_\tau = \mathbf{1} \in A_1(n_2 - n_1, m_2)$ for $> m_1$ values of $\tau \in \Omega_{n_1}$, so we are done with the base case.

Now assume $l > 1$. Let $g \in A_1(\mathbf{n}_l)$. The induction hypothesis implies $A_1(\bar{\mathbf{n}}_{l-1}) \subseteq A_1(\bar{\mathbf{n}}_l)$, where $\bar{\mathbf{n}}_l = (n_2 - n_1, m_2), \dots, (n_{l+1} - n_1, m_{l+1})$. We must show that

$$(4.6) \quad g|_\tau \in A_1((n_2 - n_1, m_2), \dots, (n_{l+1} - n_1, m_{l+1})) =: A_1(\bar{\mathbf{n}}_l)$$

for $> m_1$ values of $\tau \in \Omega_{n_1}$. By the definition of $A_1(\mathbf{n}_l)$, we have $g|_\tau \in A_1(\bar{\mathbf{n}}_{l-1})$ for $> \frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$, so the induction hypothesis implies that the inclusion (4.6) holds for the required number of $\tau \in \Omega_{n_1}$. \square

4.3. Constructing elements of $A_i(\mathbf{n}_l)$. In this subsection we estimate the cardinality of $|A_i(n, m)|$ and construct some elements of $A_i(\mathbf{n}_l)$, for the purpose of estimating $|A_i(\mathbf{n}_l)|$ in the next subsection.

Lemma 4.14. *Let $m \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \frac{|A_i(n, m)|}{|G_2^{(n)}|} = \frac{1}{2}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.*

Remark 4.15. We can also write the conclusion of Lemma 4.14 as

$$(4.7) \quad |A_i(n, m)| = |G_2^{(n)}| \left(\frac{1}{2} + o(1) \right),$$

where $o(1)$ is a quantity tending to 0 as $n \rightarrow \infty$ and m remains fixed.

Proof. Let $Z(n, m) := G_2^{(n)} \setminus (A_0(n, m) \cup A_1(n, m))$. Lemma 4.8 implies $|A_0(n, m)| = |A_1(n, m)|$ and $A_0(n, m) \cap A_1(n, m) = \emptyset$, so it suffices to show that

$$(4.8) \quad \lim_{n \rightarrow \infty} \frac{|Z(n, m)|}{|G_2^{(n)}|} = 0.$$

Now $Z(n, m) = \{g \in G_2^{(n)} : \frac{1}{2}|\Omega_n| - m \leq |g^{-1}(1)|_n \leq \frac{1}{2}|\Omega_n| + m\}$, so

$$\begin{aligned} |Z(n, m)| &\leq (2m + 1) \max_{k \in \{0, \dots, |\Omega_n|\}} \binom{|\Omega_n|}{k} \\ &= (2m + 1) \binom{|\Omega_n|}{\lfloor |\Omega_n|/2 \rfloor}. \end{aligned}$$

Since $|G_2^{(n)}| = 2^{|\Omega_n|}$ and $\binom{|\Omega_n|}{\lfloor |\Omega_n|/2 \rfloor} = o(2^{|\Omega_n|})$, we have established Equation (4.8). \square

To estimate the cardinality of $A_i(\mathbf{n}_l)$, it helps to have a “bottom up” inductive construction, somewhat dual to the “top down” inductive definition given in 4.10.

Definition 4.16. Given $g \in G_2^{(n_{l-1})}$ and $m_l \in \mathbb{N}$, let $G_2^{(n_l)}[g, m_l]$ be the set of $h \in G_2^{(n_l)}$ satisfying

$$(4.9) \quad h|_\tau \in A_{g(\tau)}(n_l - n_{l-1}, m_l) \text{ for all } \tau \in \Omega_{n_{l-1}},$$

where we abuse notation and use $g(\tau)$ to denote $i \in \mathbb{Z}/2\mathbb{Z}$ if $g([\tau]) = \{i\}$.

Lemma 4.17. *Let $l \in \mathbb{N}$, $l > 1$. Let $n_1 < \dots < n_l \in \mathbb{N}$, $m_j \in \mathbb{N}$, $i \in \mathbb{Z}/2\mathbb{Z}$ and $g \neq g' \in A_i((n_1, m_1), \dots, (n_{l-1}, m_{l-1}))$. Then*

- (i) $G_2^{(n_l)}[g, m_l] \subseteq A_i((n_1, m_1), \dots, (n_l, m_l))$,
 - (ii) $G_2^{(n_l)}[g, m_l] \cap G_2^{(n_l)}[g', m_l] = \emptyset$.
 - (iii) If $g \in G_2^{(n_{l-1})}$ and $m_l \in \mathbb{N}$ are fixed, then
- $$(4.10) \quad |G_2^{(n_l)}[g, m_l]| = |G_2^{(n_l)}| \left(\frac{1}{|G_2^{(n_{l-1})}|} + o(1) \right),$$

where $o(1)$ is a quantity tending to 0 as $n_l \rightarrow \infty$.

Proof. We prove Part (i) by induction on l . For $l = 1$ there is nothing to prove, so the base case is $l = 2$. We set $i = 1$, as the case $i = 0$ follows by symmetry. Fix $g \in A_1(n_1, m_1)$. We must prove that $G_2^{(n_2)}[g, m_2] \subseteq A_1((n_1, m_1), (n_2, m_2))$. Let $h \in G_2^{(n_2)}[g, m_2]$, so that for $\tau \in \Omega_{n_1}$, $h|_\tau \in A_1(n_2 - n_1, m_2)$ whenever $g|_\tau = \mathbf{1}$, which occurs for at least $\frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$. Then $h \in A_1((n_1, m_1), (n_2, m_2))$, by definition.

Now fix $l \in \mathbb{N}$, $l > 2$. The induction hypothesis implies that if $n'_1 < \dots < n'_{l-1}$, $m'_j \in \mathbb{N}$, and $g \in A_1((n'_1, m'_1), \dots, (n'_{l-2}, m'_{l-2}))$, then

$$G_2^{(n'_{l-1})}[g, m'_{l-1}] \in A_1((n'_1, m'_1), \dots, (n'_{l-1}, m'_{l-1})).$$

Let $g \in A_1((n_1, m_1), \dots, (n_{l-1}, m_{l-1}))$, and let $h \in G_2^{(n_l)}[g, m_l]$, so that $h|_\tau \in A_1((n_l - n_{l-1}, m_l))$ whenever $\tau \in \Omega_{n_{l-1}}$ and $g|_\tau = \mathbf{1}$. For all $\psi \in \Omega_{n_1}$, we then have $h|_\psi \in G_2^{(n_l - n_1)}[g|_\psi, m_l]$. By the induction hypothesis, the inclusion

$$(4.11) \quad g|_\psi \in A_1((n_2 - n_1, m_2), \dots, (n_{l-1} - n_1, m_{l-1}))$$

implies $h|_\psi \in A_1((n_2 - n_1, m_2), \dots, (n_l - n_1, m_l))$. Since the inclusion (4.11) occurs for at least $\frac{1}{2}|\Omega_{n_1}| + m_1$ values of $\psi \in \Omega_{n_1}$, we then have $h \in A_1((n_1, m_1), \dots, (n_l, m_l))$. This completes the induction and the proof of Part (i).

To prove Part (ii) we use Part (iv) of Lemma 4.8. Let $h \in G_2^{(n_l)}[g, m_l]$, $h' \in G_2^{(n_l)}[g', m_l]$. Since $g \neq g'$, there exists $\tau \in \Omega_{n_{l-1}}$ such that $g|_\tau \neq g'|_\tau$. Then for some $i \in \mathbb{Z}/2\mathbb{Z}$, $h|_\tau \in A_i(n_l - n_{l-1}, m_l)$ while $h'|_\tau \in A_{i+1}(n_l - n_{l-1}, m_l)$. Then $h|_\tau \neq h'|_\tau$ by disjointness of $A_i(n_l - n_{l-1}, m_l)$, $i = 0, 1$ (Part (iv) of Lemma 4.8). It follows that $h \neq h'$, as desired.

For Part (iii), we identify $G_2^{(n_l)}[g, m_l]$ with the set of functions $h : \Omega_{n_{l-1}} \rightarrow G_2^{(n_l - n_{l-1})}$ satisfying $h(\tau) \in A_{g(\tau)}(n_l - n_{l-1}, m_l)$ for $\tau \in \Omega_{n_{l-1}}$. There are

$$\prod_{\tau \in \Omega_{n_{l-1}}} |A_{g(\tau)}(n_l - n_{l-1}, m_l)|$$

such functions, so the estimate (4.7) implies

$$\begin{aligned} |G_2^{(n_l)}[g, m_l]| &= |G_2^{(n_l - n_{l-1})}|^{|\Omega_{n_{l-1}}|} \left(\frac{1}{2} + o(1)\right)^{|\Omega_{n_{l-1}}|} \\ &= |G_2^{(n_l)}| \left(\frac{1}{2^{|\Omega_{n_{l-1}}|}} + o(1)\right) \\ &= |G_2^{(n_l)}| \left(\frac{1}{|G_2^{(n_{l-1})}|} + o(1)\right), \end{aligned}$$

as desired. \square

4.4. Estimating $|A_i(\mathbf{n}_l)|$.

Lemma 4.18. *Fix $n_1 < \dots < n_{l-1} \in \mathbb{N}$ and $m_1, \dots, m_l \in \mathbb{N}$. Then for all $i \in \mathbb{Z}/2\mathbb{Z}$*

$$\liminf_{n_l \rightarrow \infty} \frac{|A_i((n_1, m_1), \dots, (n_l, m_l))|}{|G_2^{(n_l)}|} \geq \frac{|A_i((n_1, m_1), \dots, (n_{l-1}, m_{l-1}))|}{|G_2^{(n_{l-1})}|}.$$

Proof. In this proof, $o(1)$ denotes a quantity which tends to 0 as $n_l \rightarrow \infty$. We estimate the cardinality of $A_1(\mathbf{n}_l) := A_1((n_1, m_1), \dots, (n_l, m_l))$, since Lemma 4.12 implies $|A_1(\mathbf{n}_l)| = |A_0(\mathbf{n}_l)|$.

For $g \in G_2^{(n_{l-1})}$, consider $B(g) := G_2^{(n_l)}[g, m_l]$. Lemma 4.17 says that the $B(g)$ are mutually disjoint and $\bigcup_{g \in A_1(\mathbf{n}_{l-1})} B(g) \subseteq A_1(\mathbf{n}_l)$, where $A_1(\mathbf{n}_{l-1}) := A_1((n_1, m_1), \dots, (n_{l-1}, m_{l-1}))$. Part (iii) of Lemma 4.17 then implies

$$\begin{aligned} |A_1(\mathbf{n}_l)| &\geq |A_1(\mathbf{n}_{l-1})| \min_{g \in G_2^{(n_{l-1})}} |B(g)| \\ &= |A_1(\mathbf{n}_{l-1})| |G_2^{(n_l)}| \left(\frac{1}{|G_2^{(n_{l-1})}|} + o(1)\right) \quad \text{by (4.10)} \\ &= |G_2^{(n_l)}| \left(\frac{|A_1(\mathbf{n}_{l-1})|}{|G_2^{(n_{l-1})}|} + o(1)\right), \end{aligned}$$

which implies the conclusion of the lemma. \square

4.5. Proofs of Theorems. In this section we prove Theorems 1.1, 1.2, 1.4, and 1.6. The set S we construct for these theorems will be a union of some of the $V(n, k)$ defined in Section 3.5.

Lemma 4.19. *Let $(n_j)_{j \in \mathbb{N}}$ be an increasing sequence of natural numbers, and let $(m_j)_{j \in \mathbb{N}}$, $(k_j)_{j \in \mathbb{N}}$ be sequences of natural numbers. Let*

$S = \bigcup_{j=1}^{\infty} V(n_j, k_j)$, $i \in \mathbb{Z}/2\mathbb{Z}$, and $A = \bigcup_{l=1}^{\infty} A_i((n_1, m_1), \dots, (n_l, m_l))$.
Then

$$S + A \subseteq \bigcup_{l=1}^{\infty} A_{i+1}((n_1, m_1 - k_1), \dots, (n_l, m_l - k_l)).$$

Proof. As usual \mathbf{n}_l abbreviates the expression $(n_1, m_1), \dots, (n_l, m_l)$. Since $S + A = \bigcup_{l,j=1}^{\infty} V(n_j, k_j) + A_i(\mathbf{n}_l)$, it suffices to show that for each j, l ,

$$(4.12) \quad V(n_j, k_j) + A_i(\mathbf{n}_l) \subseteq A_{i+1}((n_1, m_1 - k_1), \dots, (n_r, m_r - k_r)),$$

where $r = \max(l, j)$. When $j \leq l$, the containment follows from Parts (iii) and (v) of Lemma 4.12. When $j > l$, Lemma 4.13 implies $A_i(\mathbf{n}_l) \subseteq A_i(\mathbf{n}_j)$, so that $V(n_j, k_j) + A_i(\mathbf{n}_l) \subseteq V(n_j, k_j) + A_i(\mathbf{n}_j)$, and again Parts (iii) and (v) of Lemma 4.12 imply the containment (4.12). \square

Proof of Theorems 1.2 and 1.6. Let $\varepsilon > 0$ and let $(k_j)_{j \in \mathbb{N}}$ be an increasing sequence of natural numbers. Let $m_j = 3k_j$ for each j . By Lemmas 4.14 and 4.18, we may choose an increasing sequence $(n_j)_{j \in \mathbb{N}}$ of natural numbers such that

$$(4.13) \quad \liminf_{l \rightarrow \infty} \frac{|A_1((n_1, m_1), \dots, (n_l, m_l))|}{|G_2^{(n_l)}|} \geq \frac{1}{2} - \varepsilon.$$

Let $A = \bigcup_{l=1}^{\infty} A_1(\mathbf{n}_l)$ and let $S = \bigcup_{l=1}^{\infty} V(n_l, k_l)$. Inequality (4.13) implies $d^*(A) \geq \frac{1}{2} - \varepsilon$ (see Section 2.7), and Lemma 4.1 implies every translate of S is a set of chromatic recurrence. By Lemma 2.10, we then have that $(C - C) \cap S \neq \emptyset$ whenever C is piecewise syndetic.

We will show that

$$(4.14) \quad (A - A) \cap S = \emptyset,$$

$$(4.15) \quad [(A + S) - (A + S)] \cap S = \emptyset.$$

Equation (4.14) will complete the proof of Theorem 1.2, as Lemma 2.10 implies that the existence of $g \in G$ and a piecewise syndetic $B \subseteq G$ such that $g + (B - B) \subseteq A - A$ would contradict Equation (4.14) and the fact that every translate of S is a set of chromatic recurrence.

Equation (4.15) implies that $A + S$ is not piecewise syndetic, since S has nonempty intersection with $C - C$ whenever C is piecewise syndetic.

Note that Equation (4.14) implies $A \cap (A + \mathbf{1}) = \emptyset$, as $\mathbf{1} \in S$. Setting $E := A \cup (A + \mathbf{1})$, Lemma 2.7 implies $d^*(E) > 1 - 2\varepsilon$, since $A \cap (A + \mathbf{1}) = \emptyset$ and $d^*(A) > \frac{1}{2} - \varepsilon$. Furthermore $E + S = (A + S) \cup (A + S + \mathbf{1})$. Since $A + S$ is not piecewise syndetic, $E + S$ is also not piecewise syndetic, by

the translation invariance and partition regularity of piecewise syndeticity (Lemma 5.7). Thus to prove Theorem 1.6 it suffices to establish Equation (4.15).

To prove Equation (4.14), observe that Lemma 4.19 implies

$$A + S \subseteq A' := \bigcup_{l=1}^{\infty} A_0((n_1, m_1 - k_1), \dots, (n_l, m_l - k_l)).$$

Using the fact that $m_j = 3k_j$ for each j , we find that $A' \cap A = \emptyset$, by Part (iv) of Lemma 4.12 and Lemma 4.13. We then have $(A + S) \cap A = \emptyset$, so $(A - A) \cap S = \emptyset$, completing the proof of Equation (4.14). The same argument also shows that $(A' - A') \cap S = \emptyset$, establishing Equation (4.15). \square

Proof of Theorems 1.1 and 1.4. In the proof of Theorem 1.2 we constructed $S, A \subseteq G_2$ such that $d^*(A) > \frac{1}{2} - \varepsilon$, every translate of S is a set of chromatic recurrence (and therefore a set of topological recurrence), and $(A - A) \cap S = \emptyset$. Theorem 1.4 follows from the existence of these sets and Lemma 5.8. Theorem 1.1 follows immediately from Theorem 1.4. \square

5. APPENDIX

This section contains material needed for the exposition in Section 2.

5.1. Equivalent forms. For the reader's convenience, we provide well known equivalent formulations of the properties (R_j) and (R_j^\bullet) , for $j = 4, 5, 6$, defined in Sections 2.1 and 2.2.

We need the following definition to state the next lemma.

Definition 5.1. Let G be a countable abelian group. We say that $S \subseteq G$ is a *set of measurable recurrence for group rotation G -systems* if for every such G -system (Z, m, R_ρ) and every $D \subseteq Z$ having $m(D) > 0$, there exists $g \in G$ such that $m(D \cap R_\rho^g D) > 0$.

In the following lemma \overline{E} denotes the topological closure of a subset E of a compact abelian group.

Lemma 5.2. *Let G be a countable abelian group and $S \subseteq G$. The following are equivalent.*

- (i) S satisfies (R_6) .
- (ii) If $W \subseteq G$ is a Bohr neighborhood of 0, then $S \cap W \neq \emptyset$.

- (iii) *For every homomorphism $\rho : G \rightarrow Z$, where Z is a compact abelian group, $\overline{\rho(S)}$ contains $0 \in Z$.*
- (iv) *S is a set of measurable recurrence for group rotation G -systems.*

Proof. We prove (iii) \implies (ii) \implies (i) \implies (iii), then (iv) \implies (i) and (iii) \implies (iv).

(iii) \implies (ii). Suppose S satisfies condition (iii), and let $W \subseteq G$ be a Bohr neighborhood of 0, so that there are homomorphisms $\rho_1, \dots, \rho_k : G \rightarrow \mathbb{T}$ and a neighborhood V of $0 \in \mathbb{T}$ such that W contains $\bigcap_{i=1}^k \rho_i^{-1}(V)$. Let Z be the group \mathbb{T}^k and $\rho : G \rightarrow Z$ be the homomorphism given by $\rho(g) = (\rho_1(g), \dots, \rho_k(g))$. Let $U = V \times \dots \times V \subseteq Z$, so that U is a neighborhood of $0 \in Z$. Since S satisfies condition (iii), there is a $g \in S$ such that $\rho(g) \in U$. Then $\rho_i(g) \in V$ for each i , and we conclude that $g \in W$.

(ii) \implies (i). Suppose $S \cap W \neq \emptyset$ for every Bohr neighborhood of $0 \in G$. Let (Z, R_ρ) be a minimal group rotation, and let $U \subseteq Z$ be a nonempty open set. Choose a neighborhood V of $0 \in Z$ such that $U \cap (U + v) \neq \emptyset$ for every $v \in V$, and let $W = \rho^{-1}(V)$. Then W is a Bohr neighborhood⁶ of 0, so there exists $g \in S \cap W$, meaning $\rho(g) \in V$. We then have $U \cap (U + \rho(g)) \neq \emptyset$, meaning $U \cap (R_\rho^g U) \neq \emptyset$. This shows that S satisfies (R_6) .

(i) \implies (iii). Let $\rho : G \rightarrow Z$ be a homomorphism to a compact abelian group Z and assume $S \subseteq G$ satisfies (R_6) . Assume, without loss of generality, that $\overline{\rho(G)} = Z$. Let $U \subseteq Z$ be a neighborhood of $0 \in Z$, and let $V \subseteq U$ be such that $V - V \subseteq U$. Since S satisfies R_6 , there is a $g \in S$ such that $V \cap (V + \rho(g)) \neq \emptyset$, meaning $\rho(g) \in V - V$, so $\rho(g) \in U$. Since U is an arbitrary neighborhood of $0 \in Z$, we have shown that $0 \in \overline{\rho(S)}$.

(iv) \implies (i). Suppose S satisfies condition (iv) and that (Z, R_ρ) is a minimal group rotation. Let $U \subseteq Z$ be a nonempty open set, so that $m(U) > 0$, where m is the Haar probability measure on Z . Condition (iv) implies $m(U \cap (U + \rho(g))) > 0$ for some $g \in S$, so that $U \cap (U + \rho(g)) \neq \emptyset$ for this g . Since $U \subseteq Z$ was an arbitrary open set, this shows that S satisfies (R_6) .

(iii) \implies (iv). Let (Z, m, R_ρ) be a group rotation G -system. For measurable sets $D \subseteq Z$, the map $z \mapsto m(D \cap (D + z))$ is continuous. If $m(D) > 0$, we conclude that $m(D \cap (D + z)) > 0$ for all z sufficiently

⁶Here we are using the fact that every homomorphism from G to a compact group is continuous in the Bohr topology.

close to 0, so condition (iii) implies $m(D \cap (D + \rho(g))) > 0$ for some $g \in S$. \square

See Section 2.3 for the definitions of “measure expanding,” “measure transitive,” etc.

Lemma 5.3. *Let G be a countable abelian group and $S \subseteq G$. The following are equivalent.*

- (i) S satisfies (R_6^\bullet) .
- (ii) S is dense in the Bohr topology of G .
- (iii) Every translate of S is a set of measurable recurrence for group rotation G -systems.

The following are equivalent.

- (iv) Every translate of S is a set of chromatic recurrence.
- (v) S satisfies (R_5^\bullet) .
- (vi) S is transitive for minimal G -systems.
- (vii) S is expanding for minimal G -systems.

The following are equivalent.

- (viii) Every translate of S is a set of density recurrence.
- (ix) S satisfies (R_4^\bullet) .
- (x) S is measure transitive.
- (xi) S is measure expanding.

Proof. The equivalence of (i), (ii), and (iii) is a consequence of Lemma 5.2 and the definition of the Bohr topology.

The equivalence of (iv) and (v) is due to Part (ii) of Proposition 2.9.

(v) \implies (vi). Let (X, T) be a minimal topological G -system with $U, V \subseteq X$ nonempty open sets. By the minimality of (X, T) there exists $g \in G$ such that $W := U \cap T^g V \neq \emptyset$. Since $S - g$ is a set of topological recurrence, there exists $h \in S$ such that $W \cap T^{h-g} W \neq \emptyset$, meaning

$$U \cap T^g V \cap T^{h-g}(U \cap T^g V) \neq \emptyset,$$

which implies $U \cap T^h V \neq \emptyset$. Since $U, V \subseteq X$ are arbitrary nonempty open sets, we have shown that S is transitive for minimal G -systems.

(vi) \implies (vii). Let (X, T) be a minimal topological G -system and let $x \in X$. We will show that for each nonempty open set U , there is a dense open set of points V_U such that for all $x \in V_U$, there exists $g \in S$ such that $T^g x \in U$. Fix a nonempty open set $U \subseteq X$. By condition (vi) for every nonempty open $W \subseteq X$ there exists $g \in S$ such that $T^g W \cap U \neq \emptyset$. For each open set W and each $g \in S$ let

$W_g = W \cap T^{-g}U$. Then $V_U := \bigcup_{W \subseteq X \text{ open}, g \in G} W_g$ is the desired dense open set.

Let \mathcal{U} be a countable base for the topology of X . Then $E := \bigcap_{U \in \mathcal{U}} V_U$ is a dense G_δ set such that for all $x \in E$, $\{T^g x : g \in S\}$ is dense in X .

(vii) \implies (iv). Condition (vii) is translation invariant, so it suffices to prove that if (vii) holds, then S is a set of topological recurrence. Let (X, T) be a minimal topological G -system and $U \subseteq X$ a nonempty open set. Condition (vii) permits us to choose $x \in U$ and $g \in S$ such that $T^g x \in U$. We conclude that $U \cap T^g U \neq \emptyset$ for this g . Since U is an arbitrary nonempty open set, we conclude that S is a set of topological recurrence.

The equivalence of (viii) and (ix) follows directly from Part (i) of Proposition 2.9.

(ix) \implies (x). Let $S \subseteq G$ satisfy (ix) and let (X, μ, T) be a measure preserving G -system. Let $C, D \subseteq X$ and $g \in G$ be such that $\mu(C \cap T^g D) > 0$. Let $E = C \cap T^g D$. Since $S - g$ is a set of measurable recurrence, there is an $h \in S$ such that $\mu(E \cap T^{h-g} E) > 0$. Consequently

$$\mu(C \cap T^g D \cap T^{h-g}(C \cap T^g D)) > 0,$$

which implies $\mu(C \cap T^h D) > 0$. Hence S is measure transitive.

(x) \implies (xi). Let $S \subseteq G$ satisfy condition (x) and let (X, μ, T) be an ergodic measure preserving G -system. Let $D \subseteq X$ have $\mu(D) > 0$, and let $E = \bigcup_{g \in S} T^g D$. Suppose, to get a contradiction, that $\mu(X \setminus E) > 0$. By the ergodicity of (X, μ, T) there exists $g \in S$ such that $\mu((X \setminus E) \cap T^g D) > 0$. Condition (x) then implies there exists $h \in S$ such that $\mu((X \setminus E) \cap T^h D) > 0$, a contradiction, since $T^h D \subseteq E$.

(xi) \implies (viii). Since condition (xi) is translation invariant, it suffices to prove that S is a set of density recurrence. Let $A \subseteq G$ have $d^*(A) > 0$. Then by Lemma 5.8 there is an ergodic measure preserving system (X, μ, T) and a set $D \subseteq X$ having $\mu(D) > 0$ such that $A - A$ contains $\{g \in G : \mu(D \cap T^g D) > 0\}$. Now condition (xi) implies that there exists $g \in S$ such that $\mu(D \cap T^g D) > 0$ (since otherwise $\mu(\bigcup_{g \in S} T^g D) \leq 1 - \mu(D)$) and we conclude that $(A - A) \cap S \neq \emptyset$. Since $A \subseteq G$ was an arbitrary set having $d^*(A) > 0$, we have shown that S is a set of density recurrence. \square

5.2. Minimality and shift spaces. Let G be a countable abelian group. See [15, Theorem 1.15] for a proof of the following standard lemma.

Lemma 5.4. *If (X, T) is a minimal topological G -system and $U \subseteq X$ is a nonempty open set, then for all $x \in X$ the set $\{g : T^g x \in U\}$ is syndetic.*

Consider the compact metrizable space $X = \{0, 1\}^G$. The elements of G are functions $x : G \rightarrow \{0, 1\}$. Let σ be the action of g on X defined by $(\sigma^g x)(h) = x(h + g)$ for each $g, h \in G$, $x \in X$. We call the topological G -system (X, σ) the *shift space* and σ the *shift action*. Note that a sequence of elements x_n converges to $x \in X$ if and only if for every finite $F \subseteq G$, $x_n|_F = x|_F$ for all sufficiently large n , meaning X has the topology of pointwise convergence.

Note that for $A \subseteq G$, $1_A \in X$ and for $g \in G$, $\sigma^g 1_A = 1_{A-g}$.

For the next lemma, recall that a set $A \subseteq G$ is thick if for every finite $F \subseteq G$, there is a $g \in G$ such that $F + g \subseteq A$ (Definition 2.3).

Lemma 5.5. *Let G be a countable abelian group and let (X, σ) be the corresponding shift space. Let $A \subseteq G$. Consider the following conditions.*

- (i) $A + F$ is thick for some finite set $F \subseteq G$.
- (ii) The orbit closure $\overline{\{\sigma^g 1_A : g \in G\}}$ contains a minimal subsystem not equal to $\{x_0\}$, where $x_0 \in \{0, 1\}^G$ is the constant 0 function.

Then (i) \implies (ii).

Proof. Let $F \subseteq G$ be a finite set such that $F + A$ is thick. Let $E_1 \subseteq E_2 \subseteq \dots$ be an increasing sequence of finite sets whose union is G . For each $i \in \mathbb{N}$, let g_i satisfy $E_i + g_i \subseteq F + A$. Consider the points $\sigma^{g_i} 1_A = 1_{A-g_i}$, and let $x \in X$ be a limit of these points.

Claim. *For all $g \in G$, there exists $h \in F$ such that $x(g - h) = 1$.*

Proof of Claim. Let $g \in G$, and choose N so that $g \in E_i$ for all $i \geq N$. For each $i \geq N$, we have that $A - g_i + F \supset E_i \ni g$, meaning

$$\sum_{h \in F} 1_{A-g_i}(g - h) \geq 1.$$

Then

$$(5.1) \quad 1 \leq \liminf_{i \rightarrow \infty} \sum_{h \in F} 1_{A-g_i}(g - h) \leq \sum_{h \in F} x(g - h),$$

so $x(g - h) = 1$ for some $h \in F$. □

The Claim implies that x_0 is not in the orbit closure Y of x . It follows that no minimal subsystem of X is contained in the orbit closure of x

is equal to $\{x_0\}$. Since x is in the orbit closure of 1_A , we can consider any minimal subsystem contained in Y to establish condition (ii). \square

5.3. Piecewise syndeticity. Condition (i) in the following lemma is the standard definition of “piecewise syndetic,” condition (iii) is our definition.

Lemma 5.6. *Let G be a countable abelian group and $A \subseteq G$. The following are equivalent.*

- (i) *There is a syndetic set $S \subseteq G$ and a thick set $R \subseteq G$ such that $S \cap R \subseteq A$.*
- (ii) *There is a finite set F such that $A + F$ is thick.*
- (iii) *There is a syndetic set $S \subseteq G$ such that for all finite $F \subseteq S$, there is a $g \in G$ such that $F + g \subseteq S$.*

Proof. (i) \implies (ii) Let $S, R \subseteq G$ be syndetic and thick sets, respectively, such that $S \cap R \subseteq A$. Let $F \subseteq G$ be a finite set such that $S + F = G$ and let K be an arbitrary finite subset of S .

Choose g so that $K - F + g \subseteq R$. Since $S + F = G$, we may choose elements $s_1, \dots, s_n \in S$ and $f_1, \dots, f_n \in F$ so that $K + g = \{s_i + f_i : 1 \leq i \leq n\}$. Then for each $i \leq n$, $s_i \in K - F + g$, so $s_i \in S \cap R$. It follows that $K + g \subseteq (S \cap R) + F$, so A satisfies condition (ii).

(ii) \implies (iii) Suppose $F \subseteq G$ is finite and $A + F$ is thick. Let $X = \{0, 1\}^G$ be the shift space with shift action σ , and let $1_A \in X$ be the characteristic function of A . By Lemma 5.5 there is a minimal subsystem Y contained in the orbit closure of 1_A which is not equal to $\{x_0\}$, where x_0 is the constant 0 function. If $y \in Y$, Lemma 5.4 implies the set $S := \{g : y(g) = 1\}$ is syndetic. Since y is in the orbit closure of 1_A , we have that for all finite $E \subseteq G$, there exists $g_E \in G$ such that $1_A(g) = y(g + g_E)$ for all $g \in E$. It follows that A contains $(S \cap E) - g_E$ for each finite $E \subseteq G$, so condition (iii) is satisfied.

The implication (iii) \implies (ii) is straightforward.

To prove (ii) \implies (i) we need the following definitions.

Let (X, T) be a topological G -system and d a metric on X generating the topology on X . We say that $x, y \in X$ are *proximal* if there is a sequence of elements $g_n \in G$ such that $d(T^{g_n}x, T^{g_n}y) \rightarrow 0$ as $n \rightarrow \infty$.

If $A \subseteq X$ and $x \in X$, we say that x is *proximal to A* if there is a sequence of elements $g_n \in G$ such that $\inf_{a \in A} d(T^{g_n}x, T^{g_n}a) \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 8.6 of [15] says that if A is a T -invariant closed subset of X and $x \in X$ is proximal to A , then x is proximal to some $y \in A$.

In the shift space (X, σ) , proximality of x and y is equivalent to the following condition:

(5.2) For all finite $F \subseteq G$, there is a $g \in G$ such that $x|_{F+g} = y|_{F+g}$.

We now prove (ii) \implies (i). Let $A, F \subseteq G$ be such that F is finite and $A + F$ is thick. Let (X, σ) be the shift space, and by Lemma 5.5 let $Y \subseteq X$ be a minimal subsystem of the orbit closure of 1_A such that $x_0 \notin Y$. By Proposition 8.6 of [15] we may choose $y \in Y$ such that 1_A is proximal to y . The minimality of Y implies $S := \{g : y(g) = 1\}$ is syndetic, and condition (5.2) then implies A satisfies condition (i). \square

Lemma 5.7. *Let G be a countable abelian group. If $G = \bigcup_{i=1}^k A_i$, then there is an $i \leq k$ such that A_i is piecewise syndetic. Consequently, if $A, B \subseteq G$ are not piecewise syndetic, then $A \cup B$ is not piecewise syndetic.*

For a proof, see Theorem 4.40 of [25].

5.4. Correspondence Principle. We need the following lemma to relate the properties (R_i) and (R_i^\bullet) to properties of difference sets and state various forms of the conditions (R_i) defined in Section 2.1.

Lemma 5.8. *Let G be a countable abelian group and let $A \subseteq G$ have $d^*(A) > 0$. Then there is an ergodic measure preserving G -system (X, μ, T) and a set $D \subseteq X$ with $\mu(D) \geq d^*(A)$ such that $A - A$ contains $\{g : \mu(D \cap T^g D) > 0\}$.*

Lemma 5.8 is proved for $G = \mathbb{Z}$ in Theorem 3.18 of [15]. Ergodicity is not mentioned there, but the proof is easily modified to obtain it. For an outline of a proof in the general case, see [22, Lemma 5.1].

5.5. Implications. We prove $(R_1) \implies (R_2)$ and briefly discuss the implications $(R_i) \implies (R_{i+1})$ for $i \geq 2$.

We need some tools from harmonic analysis on compact abelian groups, as presented in [34]. If G is a countable abelian group, equip G with the discrete topology, and let \widehat{G} denote the group of homomorphisms (or *characters*) $\chi : G \rightarrow \mathcal{S}^1 \subseteq \mathbb{C}$ with the topology of pointwise convergence and the group operation of pointwise multiplication. Then \widehat{G} is a compact abelian group. We write χ_0 for the identity element of \widehat{G} , which is the constant character: $\chi_0(g) = 1$ for all $g \in G$.

Lemma 5.9. $(R_1) \implies (R_2)$.

Proof. Let S satisfy (R_1) , so that we may choose a sequence $(S_j)_{j \in \mathbb{N}}$ of finite sets $S_j \subseteq S$, satisfying

$$(5.3) \quad \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \chi(g) = 0 \quad \text{for all } \chi \in \widehat{G} \setminus \{\chi_0\}.$$

Let (X, μ, T) be a measure preserving G -system and $D \subseteq X$ a measurable set. We will prove that

$$(5.4) \quad \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \mu(D \cap T^g D) \geq \mu(D)^2,$$

which implies that for all $\varepsilon > 0$ there exists $j \in \mathbb{N}$ and $g \in S_j$ such that $\mu(D \cap T^g D) > \mu(D)^2 - \varepsilon$. It therefore suffices to prove Inequality (5.4) to show that S satisfies (R_2) .

Let $f = 1_D$. Then $f \in L^2(\mu)$, and $\mu(D \cap T^g D) = \int f \cdot f \circ T^g d\mu$. The action U_T of G on $L^2(\mu)$ given by $U_T^g f = f \circ T^g$ is unitary, meaning that for each g , $U_T^g : L^2(\mu) \rightarrow L^2(\mu)$ is an invertible linear isometry. The Bochner-Herglotz theorem therefore implies the existence of a positive Borel measure σ on \widehat{G} such that $\int f \cdot f \circ T^g d\mu = \int \chi(g) d\sigma(\chi)$ for all $g \in G$. We have $\sigma(\widehat{G}) = \int 1_{\widehat{G}} d\sigma(\chi) = \int \chi(0) d\sigma(\chi) = \mu(D)$. Then

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \mu(D \cap T^g D) &= \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \int f \cdot f \circ T^g d\mu \\ &= \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \int \chi(g) d\sigma(\chi) \\ (5.5) \quad &= \int \lim_{j \rightarrow \infty} \frac{1}{|S_j|} \sum_{g \in S_j} \chi(g) d\sigma(\chi) \\ &= \int 1_{\{\chi_0\}}(\chi) d\sigma(\chi) \\ &= \sigma(\{\chi_0\}). \end{aligned}$$

The limit in (5.5) is therefore independent of the sequence $(S_j)_{j \in \mathbb{N}}$, as long as Equation (5.3) is satisfied. When $\Phi = (\Phi_j)_{j \in \mathbb{N}}$ is a Følner sequence, Φ satisfies (5.3) and the mean ergodic theorem (see [18]) implies that $\lim_{j \rightarrow \infty} \frac{1}{|\Phi_j|} \sum_{g \in \Phi_j} f \circ T^g = Pf$, where Pf is the orthogonal projection of f onto the closed space of T -invariant functions in $L^2(\mu)$.

By (5.5), we then have

$$\begin{aligned}
\sigma(\{\chi_0\}) &= \int f \cdot Pf \, d\mu \\
&= \int Pf \cdot Pf \, d\mu && \text{since } P \text{ is an orthogonal projection} \\
&\geq \left(\int Pf \, d\mu \right)^2 && t \mapsto t^2 \text{ is convex} \\
&= \left(\int f \, d\mu \right)^2 && P1_X = 1_X \\
&= \mu(D)^2 && f = 1_D,
\end{aligned}$$

so $\sigma(\{\chi_0\}) \geq \mu(D)^2$, and (5.5) then implies Inequality (5.4). \square

The implications $(R_2) \implies (R_3)$ and $(R_3) \implies (R_4)$ are straightforward. The implication $(R_4) \implies (R_5)$ is a consequence of the Bogoliouboff-Kryloff Theorem: every topological G -system (X, T) admits a T -invariant probability measure. Consequently, every minimal topological G -system (X, T) admits a T -invariant probability measure μ having full support, since the support of a T -invariant measure is a T -invariant compact subset of X . It follows that every nonempty open set $U \subseteq X$ has $\mu(U) > 0$, and then the fact that S satisfies (R_4) implies that there is a $g \in S$ such that $U \cap T^g U \neq \emptyset$.

The implication $(R_5) \implies (R_6)$ is straightforward.

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